Chapter 1 Introduction

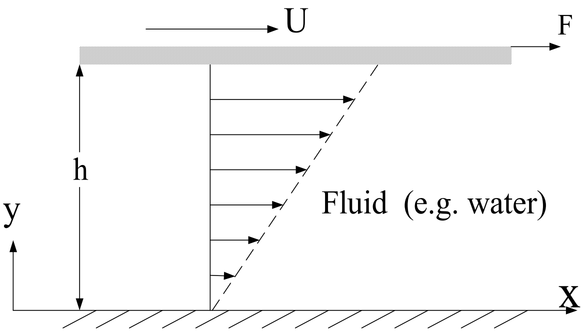
1.1 Classification of a Fluid (A fluid can only substain tangential force when it moves)

1.) By viscous effect: inviscid & Viscous Fluid.
2.) By compressible: incompressible & Compressible Fluid.
3.) By Mack No: Subsonic, transonic, Supersonic, and hypersonic flow.
4.) By eddy effect: Laminar, Transition and Turbulent Flow.

The objective of this course is to examine the effect of tangential (shearing) stresses on a fluid.

Remark:
For a ideal (or inviscid) flow, there is only normal force but tangential force between two contacting layers.

1.2 Simple Notation of Viscosity



(tangential force required to move upper plate at velocity of U)

u(y) = y/h U

From observation, the tangential force per unit area required is proportional to U/h, or du/dy. Therefore

$$\tau \equiv \text{shear stress} = \text{tangential force per unit area (F/A)} \propto \frac{U}{h}$$

or

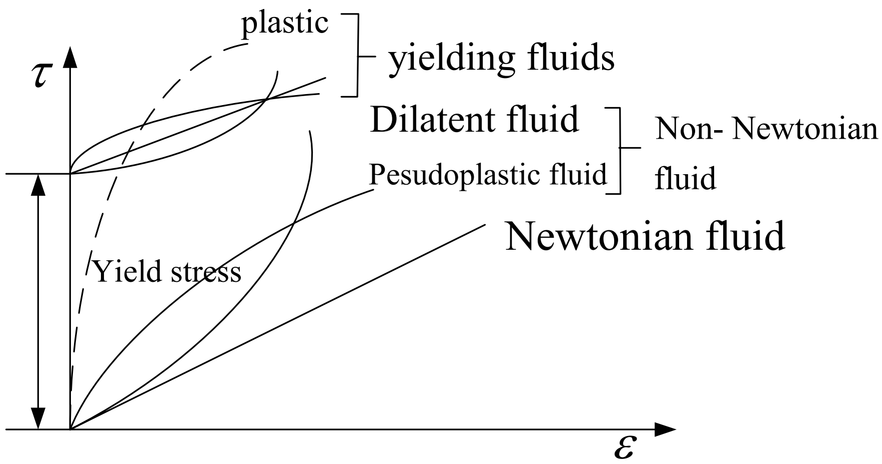
$$\tau = \mu \frac{U}{h} = \boxed{\mu \frac{\partial u}{\partial y}} \qquad \text{〝Newton's Law of function〞} \qquad (1.1)$$

μ : Constant of proportionality
The first coefficient of viscosity

Remark:
E.g. (1.1) provides the definition of the viscosity and is a method for measuring the viscosity of the fluid.

In generally, if ε_{XY} represent the strain rate, then

$$\tau_{xy} = f(\varepsilon_{xy}) \tag{1.2}$$



Newtonian fluid: linear relation between τ and ε

Pesudoplastic fluid: the slope of the curve decrease as ε increase (shear-thinning) of the shear-thinning effect is very strong. The fluid is called plastic fluid.

Dilatent fluid: the slope of the curve increases as ε increases (shear-thicking).

Yielding fluid: A material, part solid and part fluid can substain certain stresses before it starts to deform.

Note

1 Pa (Pascal) ≡ $\frac{Newton}{m^2}$ (Pascal, a French philosopher and Mathematist)

(a unit of pressure)

[μ] = [pa · sec] $\left(= \frac{kg \cdot m/s^2}{m^2} \cdot s = \frac{kg}{m \cdot s} = 10 \frac{g}{cm \cdot s}\right)$

The metric unit of viscosity is called the poise (p) in honor of J.L.M. Poiseuille (1840), who conducted pioneering experiment on viscous flow in tubes.

1 P ≡ $\frac{1g}{(cm)(s)} = 0.1 \quad pa \cdot \sec$

The unit of viscosity:

$$[\mu] = \left[\frac{\tau}{\alpha u / \partial y} \right] = \left[\frac{F / L^2}{\frac{L}{T} / L} \right] = \left[\frac{F}{L^2} T \right] \qquad \leftarrow \text{(Old -English Unit: F-L-T)}$$

or

$$[\mu] = \left[\frac{ML / T^2}{L^2} \cdot T \right] = \left[\frac{M}{LT} \right] \qquad \leftarrow \text{(international system SI unit: M-L-T)}$$

$\boxed{\text{Denote}}$: $\frac{N}{M^2} \equiv$ Pa, then

$$\begin{cases} \mu_{water,20^\circ c} = 1.01 \times 10^3 \, Pa \cdot \sec \\ \mu_{water,100^\circ c} = 283 Pa \cdot \sec \end{cases} \qquad \text{(liquid): T} \nearrow \ \rightarrow \ \mu \searrow$$

$$\begin{cases} \mu_{air,20^\circ c} = 17.9 Pa \cdot \sec \\ \mu_{air,100^\circ c} = 22.9 Pa \cdot \sec \end{cases} \qquad \text{(gas): T} \nearrow \ \rightarrow \ \mu \nearrow$$

For dilute gas:

$$\frac{\mu}{\mu_\circ} \approx \left(\frac{T}{T_\circ} \right)^n \qquad \text{(Power- law)}$$

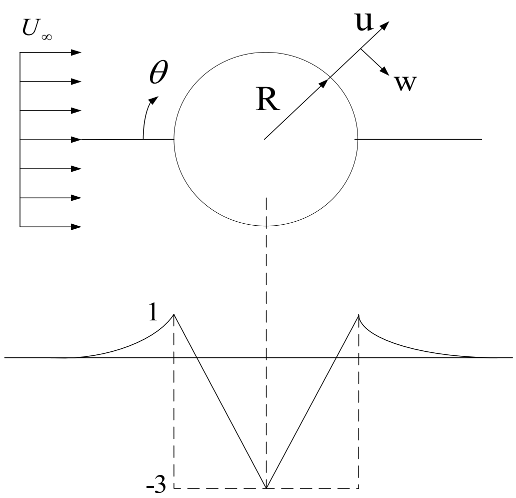
$$\frac{\mu}{\mu_\circ} \approx \left(\frac{T}{T_\circ} \right)^{3/2} \frac{T_\circ + S}{T + S} \qquad \text{(Sutherland's law)}$$

Where μ_0, T_0 and S depends on the nature of the gases.

Kinematics Viscosity $\upsilon \equiv \frac{\mu}{\rho}$

Exp: (Effect of Viscosity on fluid)

<u>Flow past a cylinder</u>



Foe a ideal flow:

$$u(r,\theta)=U_\infty \cos\theta\left(\frac{R^2}{r^2}-1\right)$$

$$v(r,\theta)=U_\infty \sin\theta\left(\frac{R^2}{r^2}+1\right)$$

At r = R, u=0, $v=2U_\infty \sin\theta$

The Bernoulli e.g. along the surface is:

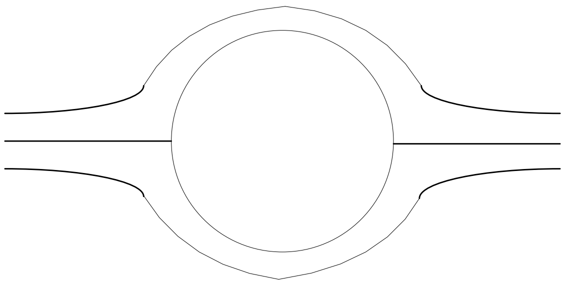
$$\frac{1}{2}\rho U_\infty^2+P_\infty=\frac{1}{2}\rho v^2+p \qquad \text{(Incompressible flow)}$$

$$C_p=\frac{P-P_\infty}{\frac{1}{2}\rho U_\infty^2}=1-\frac{v^2}{U_\infty^2}=1-\frac{1}{4}\sin^2\theta$$

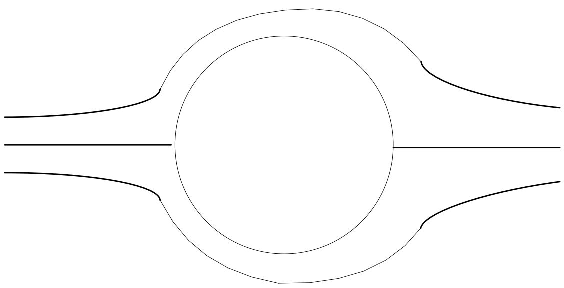
D'Albert paradox: No Drag.

For a real flow: (viscous effect in)

$$\mathrm{Re}=\frac{\rho VD}{\mu}$$

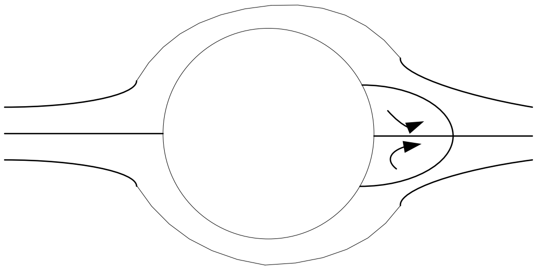


(前後幾乎對稱) Re=0.16 (fig 6)



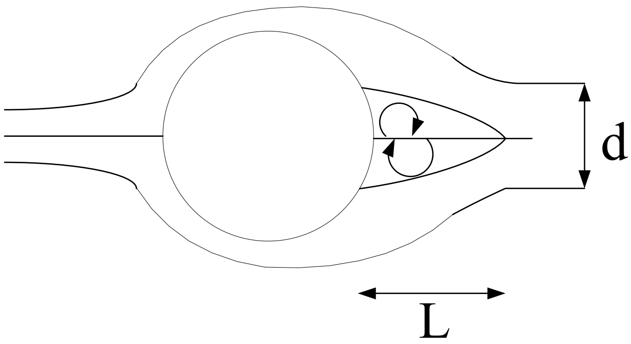
(前後不對稱) Re=1.54 (fig 24)

Separation occur



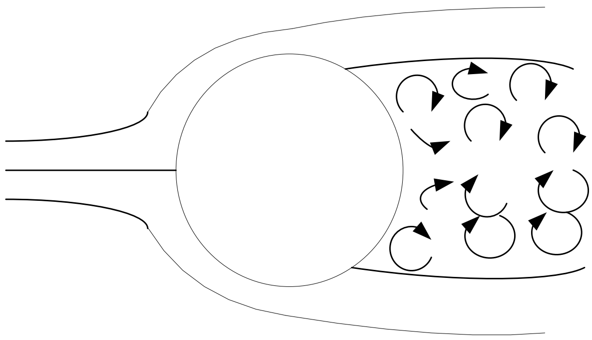
(pair of recirculating eddies)

Re=9.6 (fig 40) (6 < Re <40)



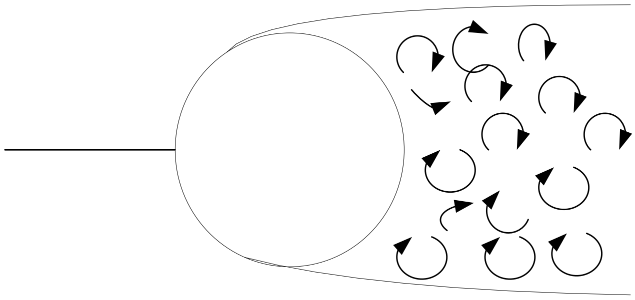
$d \sim \sqrt{\mathrm{Re}}$ R=26 (fig. 26)

$L \sim \mathrm{Re}$



supercritical

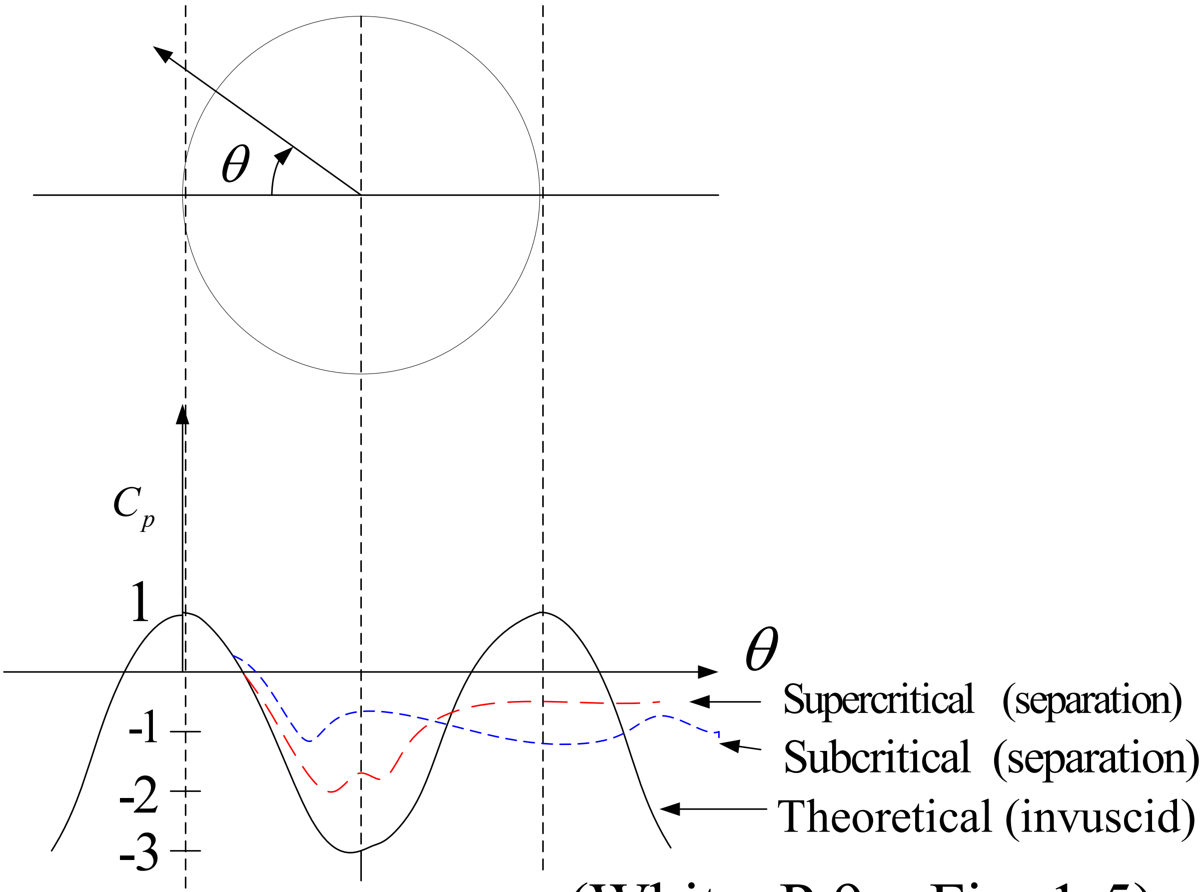
$\theta > 90°$



Subcritical

$\theta_{sep} < 90°$

The pressure distribution then becomes:



(White. P.9. Fig. 1-5)

Remark:

Newtonian Fluid Non-Newton Fluid

For a Newtonian fluid:

$\overset{\leftrightarrow}{\tau} = -\mu \overset{\leftrightarrow}{\varepsilon}$ $\quad\overset{\leftrightarrow}{\tau}$: stress tension

$\overset{\leftrightarrow}{\varepsilon}$: rate of strain tension

μ = a constant for a given temp, pressure and composition

Lf μ is not a constant for a given temp, pressure and composition, then the fluid is called *Non-Newtonian fluid*. The Non-Newtonian fluid can be classified into several kinds depending on how we model the viscosity. For example:

(I) Generalized Newtonian fluid

$\overset{\leftrightarrow}{\tau} = -\eta \overset{\leftrightarrow}{\varepsilon}$ $\quad\eta$ = a function of the scalar invariants of $\overset{\leftrightarrow}{\varepsilon}$

(i) The Carreau-Yusuda Model

$$\frac{\eta-\eta_\infty}{\eta_0-\eta_\infty}=[1+(\lambda\varepsilon)^a]^{\frac{(n-1)}{a}}$$

ε: magnitude of the $\overset{\leftrightarrow}{\varepsilon}$

(ii) power-Law model

$$\eta = m\varepsilon^{n-1}$$

$$\begin{cases} \text{n<1: pseudo plastic (shear thinning)} \\ \text{n=1: Newtonian fluid} \\ \text{n>1: dilatant (shear thickening)} \end{cases}$$

(II) Linear Viscoelastic Fluid
(III) Non-linear Viscoelastic Fluid
} → polymeric fluids

→ The fluid has both 〝viscous〞 and 〝elastic〞 properties.

By 〝elasticity〞 one usually means the ability of a material to return to some unique, original shape on the other hand, by a 〝fluid〞, one means a material that will take the shape of any container in which it is left, and thus does not possess a unique, original shape. Therefore the viscoelastic fluid is often returned as 〝memory fluid〞.

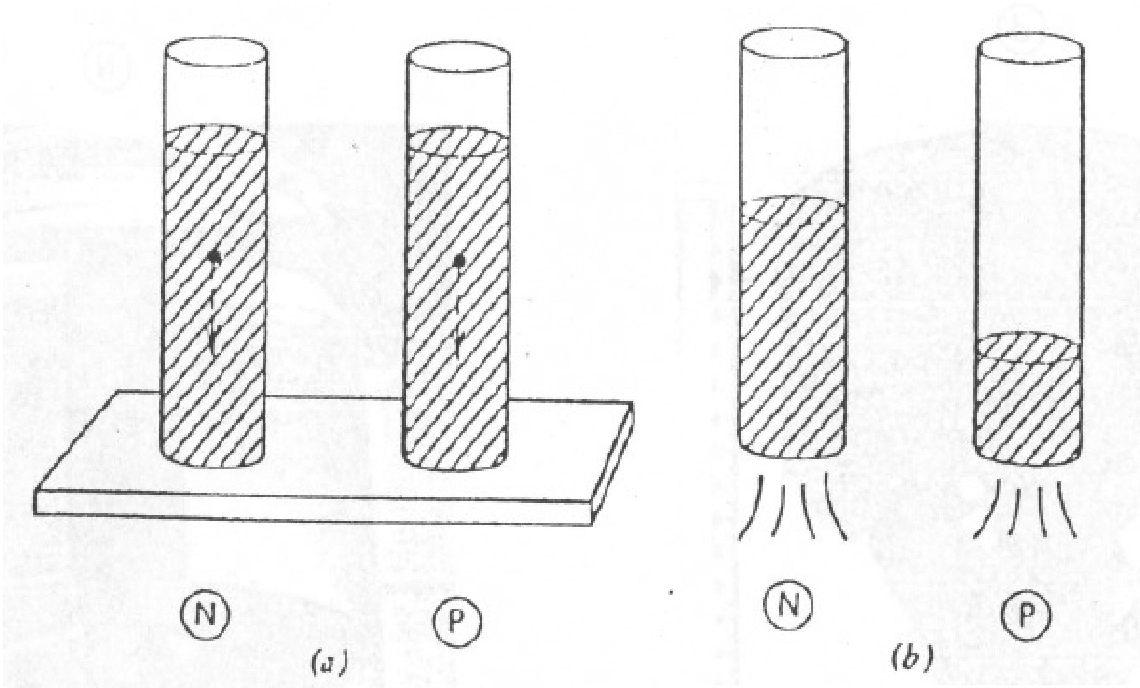


FIGURE 2.2- 1 Tube flow and “shear thinning.” In each part, the Newtonian behavior is shown on the left (N) ; the behavior of a polymer on the right (P). (a) A tiny sphere falls at the same through each; (b) the polymer out faster than Newtonian fluid.

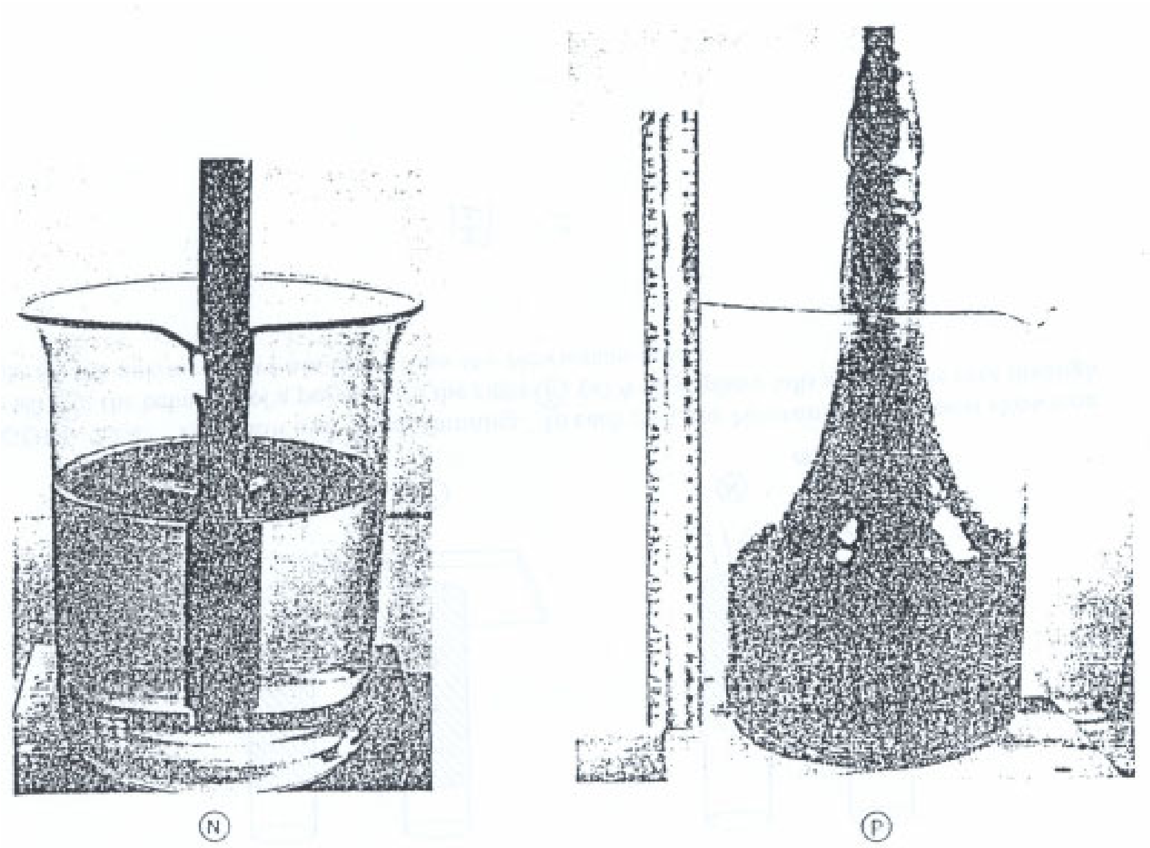
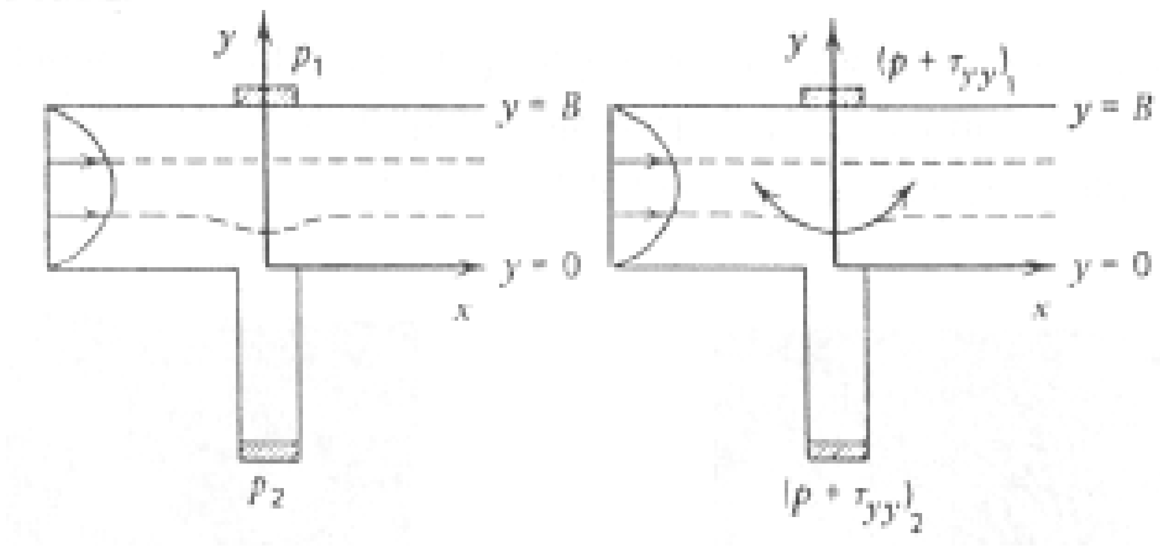


FIGURE2.3-1. fixed cylinder with rotating rod N. The Newtonian liquid, glycerin, shows a vortex; P the polymer solution, polyacrylamide in glycerin, climbs the rod. The rod is rotated much faster in the glycerin than in the polyacrylamide solution. At comparable low rates of rotation of the shaft, the polymer will climb whereas the free surface of the Newtonian liquid will remain flat. [Photographs courtesy of Dr. F. Nazem, Rheology Research Center, University of Wisconsin- Madison.]



$P_1 = P_2$ $\qquad\qquad$ $P^* = (P + \tau_{yy})_1 - (P + \tau_{yy})_2 > 0$

FIGURE 2.3-4 A fluid is flowing from left to right between two parallel plates across a deep transverse slot. "Pressure" are measured by flush-mounted transducer "1." and recessed transducer "2." (N) For the Newtonian fluid $P_1=P_2$. (P) For polymer fluids $(P+ \tau_{yy})_1 > (P+ \tau_{yy})_2$. The arrows tangent to the streamline indicate how the extra tension along a streamline tends to "lift" the fluid out of the holes, so that the recessed transducer gives a reading that is lower than that of the flush mounted transducer.

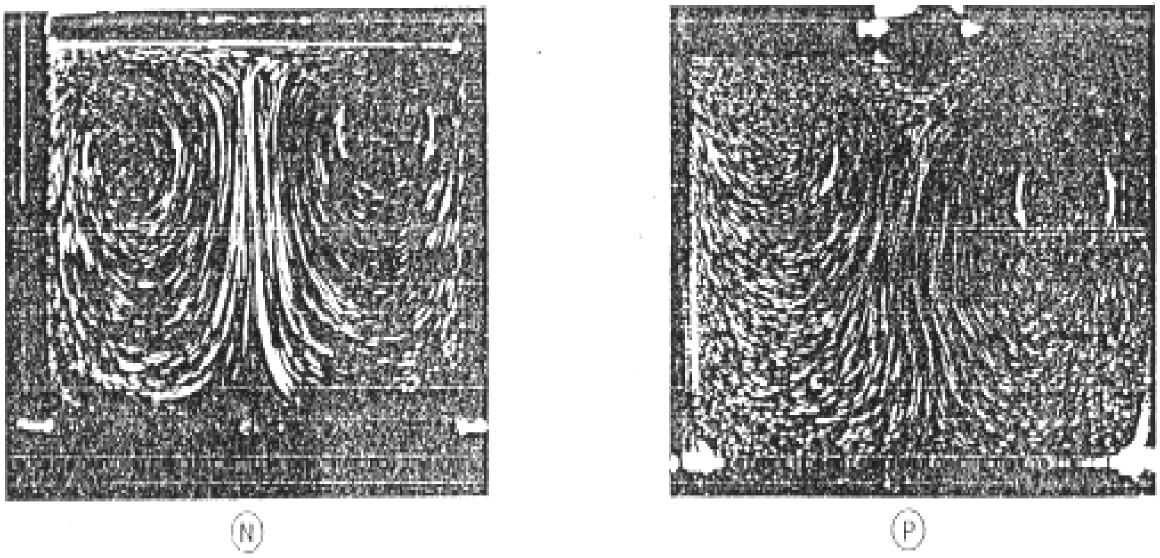


FIGURE 2.4-2 Secondary flows in the disk-cylinder system. N The Newtonian fluid moves up at the center, whereas P the viscoelastic fluid , polyacrlamid (Separan 30)-glycerol-water, moves down at the center. [Reproduced from C. T. Hill, Trans. Soc. Rheol , 213-245 (1972).]

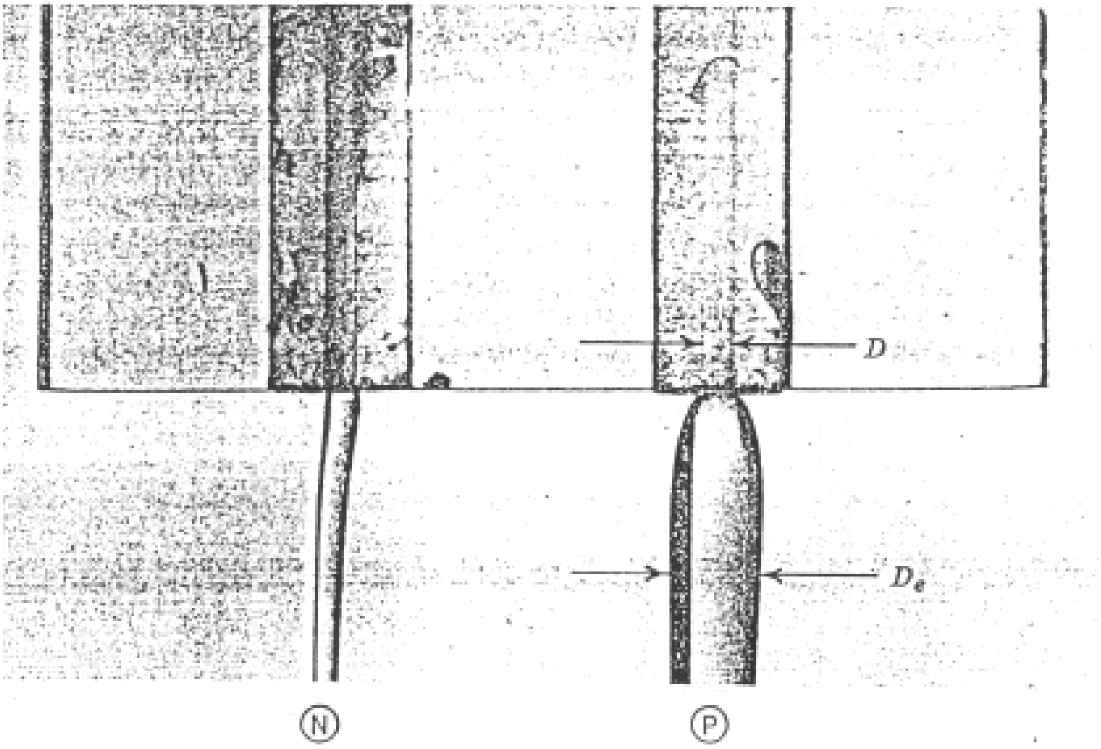


FIGURE 2.5-1 Behavior of fluids issuing from orifices. N A stream of Newtonian fluid (silicone fluid) shows no diameter increase upon emergence from the capillary tube ; P a solution of 2.44g of polymethylmethacrylate ($\bar{M} = 10^6 \, g/mol$) in 100 cm^3 of dimethylphthalate shows an increase by a factor of 3 in diameter as it flows downward out of the tube. [Reproduced from A. S. Lodge, Elastic Liquid, Academic Press, New York (1964), p. 242.]

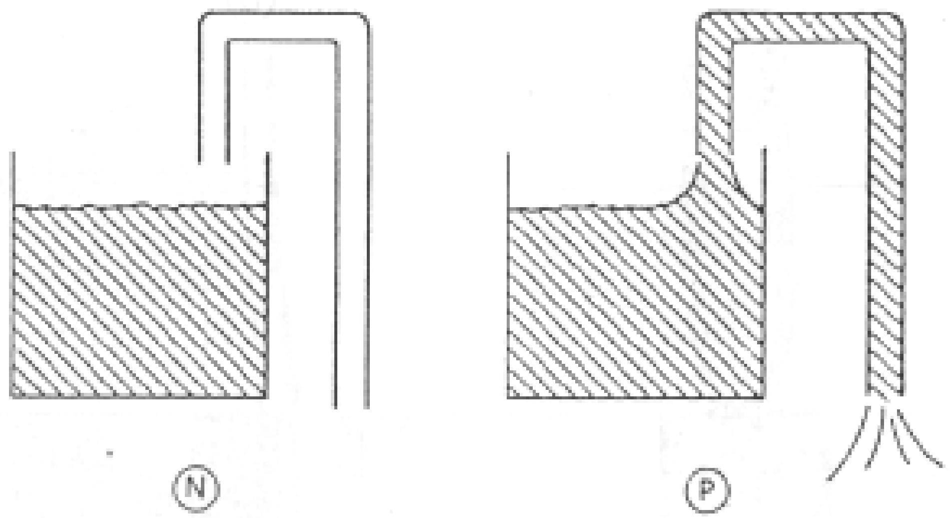


FIGURE 2.5-2 the tubeless siphon. N When the siphon tube is lifted out of the fluid, the Newtonian liquid stops flowing; P the macromolecular fluid continues to be siphoned.

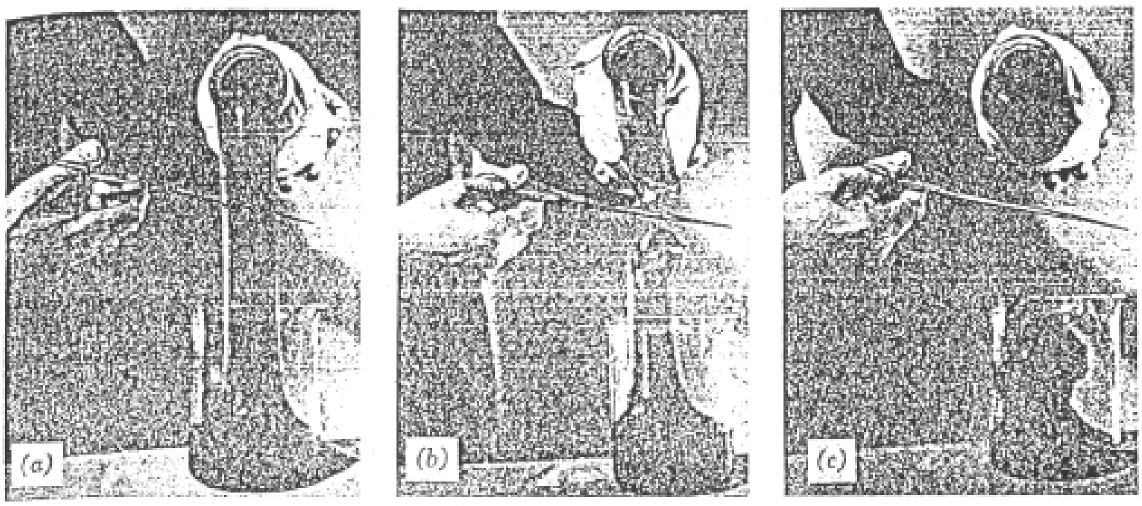
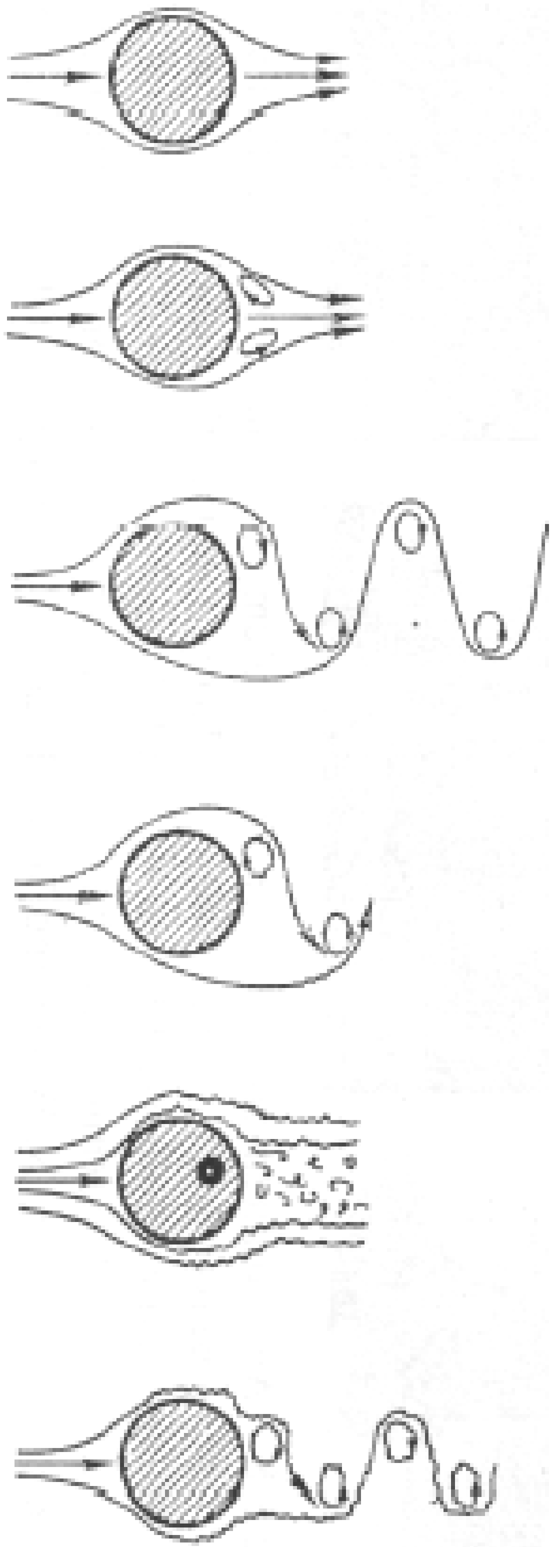


FIGURE 2.5-8 AN aluminum soap solution, made of aluminum dilaurate in decalin and m-cresol, is (a) poured from a beaker and (b) cut in midstream. In (c), note that the liquid above the cut springs back to the beaker and only the fluid below the cut falls to the container.[Reproduced from A. S. Lodge, Elastic liquids, Academic Press, New York (1964), p. 238. For a further discussion of aluminum soap solutions see N. Weber and W. H. Bauer, J. Phys. Chem., 60, 270-273 (1956).]



$Re_D < 5$ 無分離流動

5 到 $15 \le Re_D < 40$

渦脊中具 Foppl 渦旋

$4 \le Re_D < 90$ 和 $90 \le Re_D < 150$

渦旋串（Vortex street）為層流

$150 \le Re_D < 300$
$300 \le Re_D < 3\times10^5$

$3\times10^5 < Re_D < 3.5\times10^6$

層流邊界層變成紊流

$3.5\times10^6 \le Re_D < \infty(?)$

完全紊流邊界層

圖 7-6 正交流過圓柱之流動情形

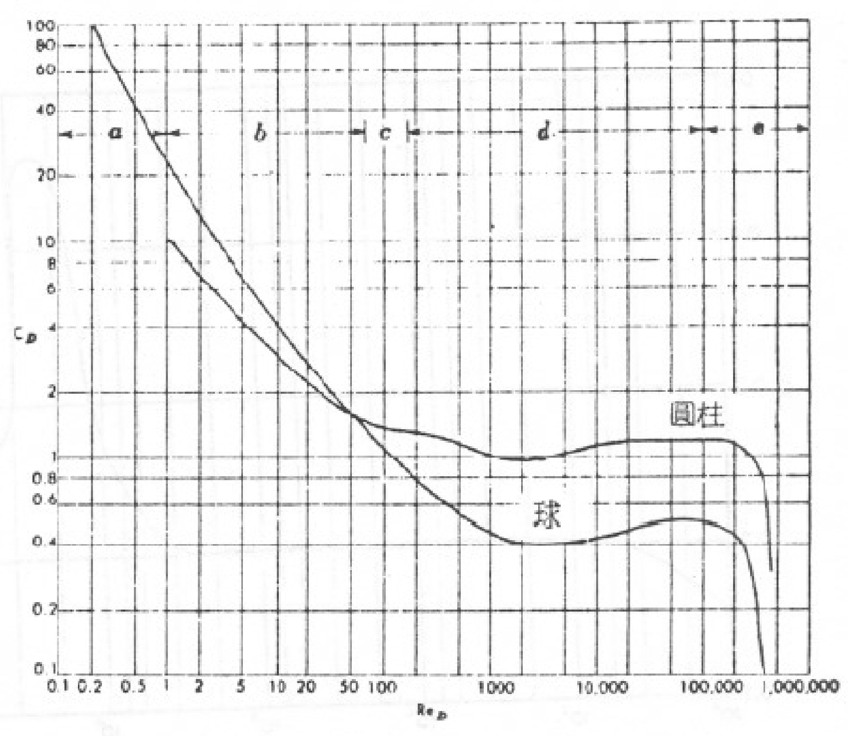


圖 7-7 長圓柱及球體之阻力係數 C_p 與 Re 數關係

以下討論不同雷諾數下的物理現象：

（1） 雷諾數的數量級為 1 或更小時，流場沒有分離現象，黏滯力是阻力的唯一因素，此時流場可由勢流理論(Potential flow theory)來導證，在圖 7-7 中阻力係數隨著雷諾數的提高而直線變化下降。

（2） 雷諾數的數量級為 10 時，流場漸漸發生渦流，在圓柱後面有小渦旋(Vertex) 出現，此時阻力的因素除了邊界層阻力外尚有渦流的因素，阻力係數依雷諾數的提高而下降。

（3） 雷諾數介於 40 到 150 之間時，圓柱後形成渦旋串(Vertex street），產生渦旋的頻率 f_v 與流場雷諾數大小有關，定義 Strouhal 數 Sh：

$$Sh = \frac{f_v \cdot D}{u_\infty} \tag{7-32}$$

Sh 與雷諾數 Re_D 的關係如下圖 7-8；此時阻力主要由係由渦流造成。

（4） 雷諾數介於 150～300 時，渦流串由層性漸漸轉變成紊性，雷諾數 300 到

3×10^5 之間，渦旋串是完全紊性的，流場非二次元性質，必須三次元才能完全描述流場，分離點的位置變化不大，由 $\theta=80^\circ$ 到 $\theta=85^\circ$，且自圓柱前端到分離點的流場維持屬性，所以此時阻力係數也幾乎維持在固定值。雷諾數不影響阻力係數也就是說黏滯力對阻力的影響很小。

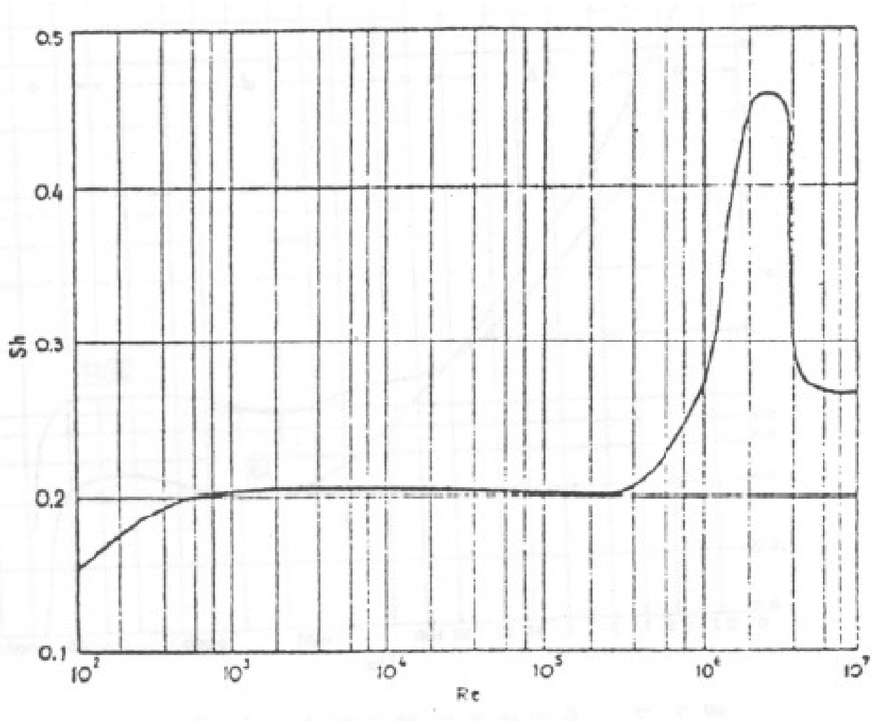


圖 7-8 Sh 數與 Re 數關係

（5） 雷諾數大於 3×10^5，阻力係數 C_D 急遽下降，也就是阻力下降，這是因為分離點往圓柱後面移動的緣故，見圖 7-9[7]；分離點會再往前移。此時阻力係數會回升。渦脊變窄無次序，不再出現渦流串。阻力的形成可分為兩個因數，邊界層存在時沿邊界層的地方有黏滯阻力存在，分離點之後面產生渦流，這是低壓地區，以致有反流的現象，造成圓柱前後壓力不平衡，是阻力產生的最大原因；當 $Re=10^6$ 之後流場本身穩定性不足以維持流場的穩定，分離點再度向前移，使阻力係數再回升；這些流場現象不僅影響到阻力，也影響到對流熱傳係數。

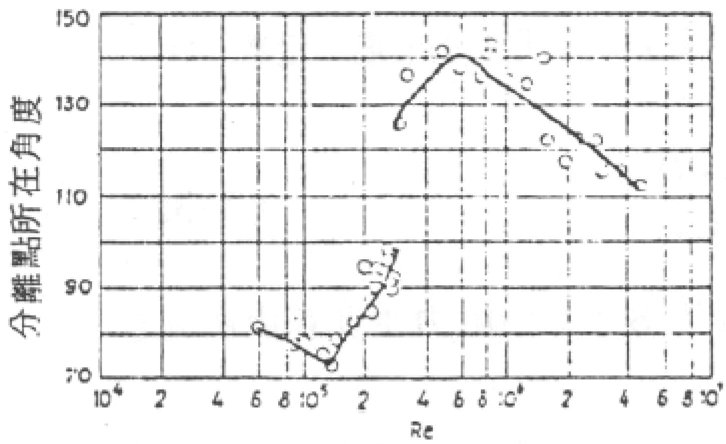


圖 7-9 管之分離點位置與 Re 數關係[7]

(二)熱傳係數

（1） 圓柱四周流場變化多端，要求得各點熱傳係數的解析值是很困難的，圖 7-10 是 W. H. Giedt [8] 所做的實驗結果，當雷諾數大於 1.4×10^5 之後，熱傳係數 $Nu(\theta)$ 出現兩個最低點，第一個最低點是邊界層由層性過渡到紊性時發生，第二個最低點則是由於分離現像。

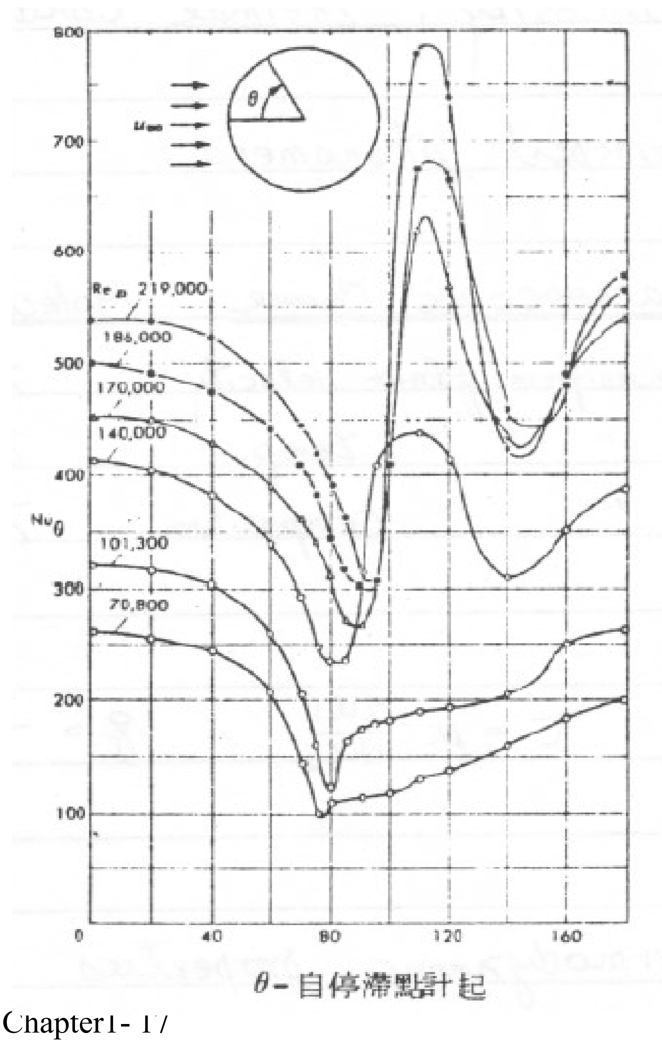


圖 7-10 圓柱之 Nu 與自停滯點計起角度 θ 關係

1. 3 Properties of Fluids

There are four types of properties:

Property is a point function,
not a point function.

1. Kinematic properties

   (Linear velocity, angular velocity, vorticity, acceleration, stain, etc.)

   —strictly speaking, these are properties of the flow field itself rather than of the fluid.

2. Transport properties

   (Viscosity, thermal conductivity, mass diffusivity)

   Transport phenomena:

Macroscopic cause	*Molecular Transport*	*Macroscopic Reset*
Non uniform flow velocity	Momentum	Viscosity
Non uniform flow temp	Energy	Heat conduction
Non uniform flow composition	Mass	Diffusion

   e.g.: $\tau = \mu \dfrac{du_1}{du_2}$, $g = -K \dfrac{dT}{dx_2}$, $\Gamma_A = -D_{AB} \dfrac{dx_A}{dx_2}$

3. Thermodynamic properties

   (pressure, density, temp, enthalpy, entropy, specific heat, prandtl number, bulk modulus, etc)

   —Classical thermodynamic, strictly speaking, does not apply to this subject, since a viscous fluid in motion is technically not in equilibrium. However, deviations from local thermodynamic equilibrium are usually not significient except when flow residence time are short and the number of molecular particles, e.g., hypersonic flow of a rarefied gas.

4. Other miscellaneous properties

   (surface tension, vapor pressure, eddy-diffusion coeff, surface-accommodation coefficients, etc.)

1.4 Boundary Conditions

(1) Fluid In permeable solid interface

(i) No slip: $\vec{V}_{fluid} = \vec{V}_{solid}$

(ii) No temperature jump: $T_{fluid} = T_{sol}$ (when the thermal contact between solid-fluid is good, i.e. $B_i = \dfrac{hL}{k} >> 1$)

or equality of heat flux (when the solid heat flux is known)

$$(K\frac{\partial T}{\partial n})_{fluid} = q \quad \text{(from solid to fluid)}$$

Remark:

If fluid is a gas with large mean free path (Normally in high Mach number & low Reynolds No.), there will is velocity jump and temperature jump in the interface.

(2) Fluid-permeable Wall interface

$$\begin{cases} (V_t)_{fluid} = (V_t)_{wall} & \text{(no slip)} \\ (V_n)_{fluid} \neq (V_n)_{wall} & \text{(flow through the wall)} \end{cases}$$

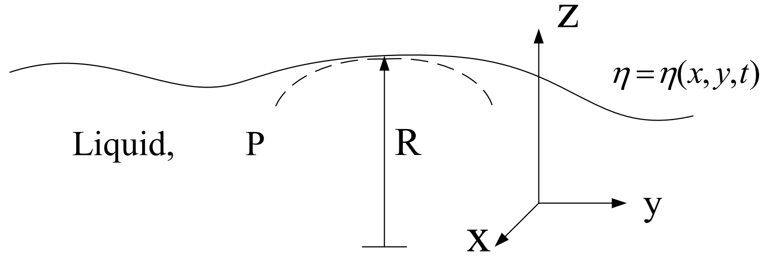
$$\begin{cases} T_{fluid} = T_{wall} & \text{(Suction)} \\ k\dfrac{dT}{dn}|_{wall} \approx \rho_{fluid} V_n C_p (T_{wall} - T_{qluid}) & \text{(injection)} \end{cases}$$

Remark:

(1) $\rho_{fluid} V_n$ is the mass flow of coolant per unit area through the wall. The actual numerical value of V_n depends largely the pressure drop across the wall. For example: Darcy's Law given $\vec{V} = -\dfrac{\overleftrightarrow{k}}{\mu} \cdot \nabla p$

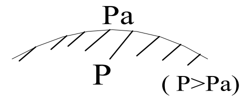
$$\text{or} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = -\frac{1}{\mu} \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix} \begin{bmatrix} \partial p/\partial x \\ \partial p/\partial y \\ \partial p/\partial z \end{bmatrix}$$

(3) Free liquid Surface

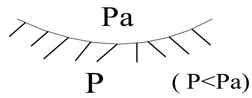


(i) At the surface, particles upward velocity (w) is equal to the motion of the free surface w(x, y, z) $= \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}$

(ii) Pressure difference between fluid & atmosphere is balanced by the surface tension of the surface.



$$P(x, y, \eta) = P_a - \sigma\left(\frac{1}{R_x} + \frac{1}{R_y}\right)$$

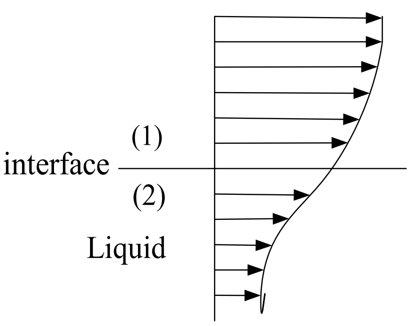


<u>Remark</u>:

In large scale problem, such as open-channel or river flow, the free surface deforms only slightly and surface-tension effect are negligible, therefore

$$W \approx \frac{\partial \eta}{\partial t}, \qquad P \approx P_a$$

（4）Liquid-Vapor or Liquid-Liquid Interface

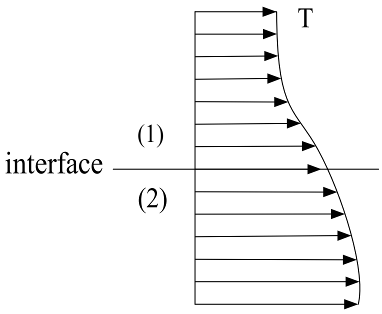


$\vec{V_1} = \vec{V_2}$ （$V_{n1} = V_{n2}$, $\vec{V_{t1}} = \vec{V_{t2}}$ ）

$P_1 = P_2$ (if surface tension is neglected)

$\tau_1 = \tau_2$ —(*)

or $\mu_1 \frac{\partial V_{t1}}{\partial n} = \mu_2 \frac{\partial V_{t2}}{\partial n}$, this is the slope $\frac{\partial V_t}{\partial n}$ need not be equal



$T_1 = T_2$

$q_1 = q_2$ (Since interface has vanishing mass,

it can't store momentum or energy.)

$$\left[\text{or} \quad -k_1 \frac{\partial T_1}{\partial n} = -k_2 \frac{\partial T_2}{\partial n} \right] \qquad \text{—(**)}$$

<u>Remark</u>:

(1) If region (1) is vapor, its μ & k are usually much smaller than for a liquid, therefore, we may approximate E.g. (*) & (**) as

$$\left(\frac{\partial V_t}{\partial n}\right)_{liq} \approx 0 \quad , \quad \left(\frac{\partial T}{\partial n}\right)_{liq} \approx 0$$

(2) If there is evaporation, condensation, or diffusion at the interface, the mass flow must be balance, $\dot{m}_1 = \dot{m}_2$.

$$D_1 \frac{\partial C_1}{\partial n} = D_2 \frac{\partial C_2}{\partial n}$$

(5) Inlet and Exit Boundary Conditions

For the majority of viscous-flow analysis, we need to know $\vec{V}$, P, and T at every point on <u>inlet & exit</u> section of the flow. However, through some approximation or simplification, we can reduce the boundary condition s needed at <u>exit</u>.

Supplementary Remarks

(1) Transports of momentum, energy, and mass are often similar and sometimes genuinely analogous. The analogy fails in multidimensional problems become heat and mass flux are vectors while momentum flux is a tension.

(2) Viscosity represents the ability of a fluid to flow freely. SAE30 means that 60 ml of this oil at a specific temperature takes 30s to run out of a 1.76 cm hole in the bottom of a cup.

(3) The flow of a viscous liquid out of the bottom of a cup is a difficult problem for which no analytic solution exits at present.

(4) For some non-Newtonian flow, the shear stress may vary w.r.t time as the strain rate is held constant, and vice versa.

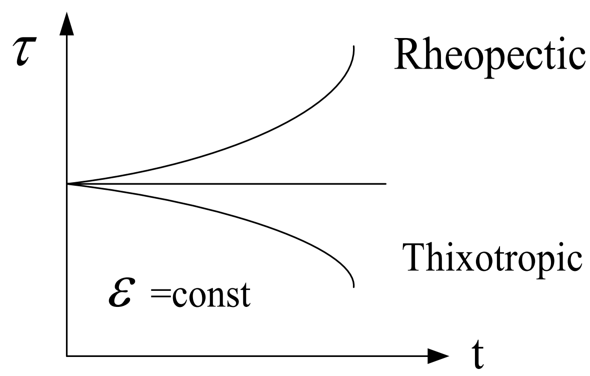


Figure 1.8. Leonardo: *Old man and Vortices;* probably a self-portrait (Windsor Castle, Royal Library, copyright reserved).

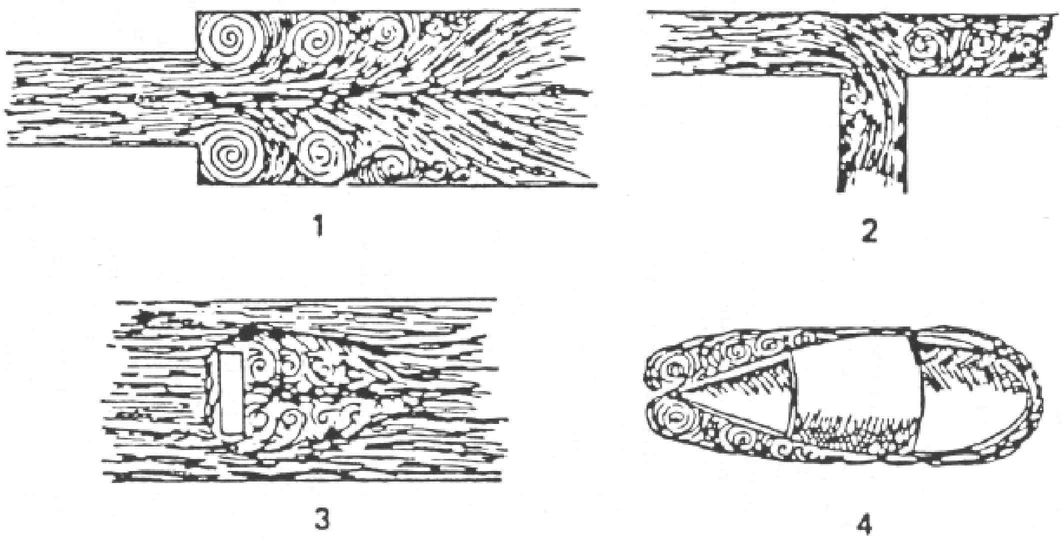


Figure 1.6. Sketches by Leonardo (from *Handbuch der Experimentalphysik* [20]).

TABLE 1.9 Chronological Listing of Some Contributors to the Science of Fluid Mechanics Noted in Text[a]

ARCHIMEDES (287–212 B.C.)
Established elementary principles of buoyancy and flotation.

SEXTUS JULIUS FRONTINUS (40–103 A.D.)
Wrote treatise on Roman methods of water distribution.

LEONARDO da VINCI (1452–1519)
Expressed elementary principle of continuity; observed and sketched many basic flow phenomena; suggested designs for hydraulic machinery.

GALILEO GALILEI (1564–1642)
Indirectly stimulated experimental hydraulics; revised Aristotelian concept of vacuum.

EVANGELISTA TORRICELLI (1608–1647)
Related barometric height to weight of atmosphere, and form of liquid jet to trajectory of free fall.

BLAISE PASCAL (1623–1662)
Finally clarified principles of barometer, hydraulic press, and pressure transmissibility.

ISAAC NEWTON (1642–1727)
Explored various aspects of fluid resistance—inertial, viscous, and wave; discovered jet contraction.

HENRI de PITOT (1695–1771)
Constructed double-tube device to indicate water velocity through differential head.

DANIEL BERNOULLI (1700–1782)
Experimented and wrote on many phases of fluid motion, coining name "hydrodynamics"; devised manometry technique and adapted primitive energy principle to explain velocity–head indication; proposed jet propulsion.

LEONHARD EULER (1707–1783)
First explained role of pressure in fluid flow; formulated basic equations of motion and so-called Bernoulli theorem; introduced concept of cavitation, and principle of centrifugal machinery.

JEAN le ROND d'ALEMBERT (1717–1783)
Originated notion of velocity and acceleration components, differential expression of continuity, and paradox of zero resistance to steady nonuniform motion.

ANTOINE CHEZY (1718–1798)
Formulated similarity parameter for predicting flow characteristics of one channel from measurements on another.

GIOVANNI BATTISTA VENTURI (1746–1822)
Performed tests on various forms of mouthpieces—in particular, conical contractions and expansions.

LOUIS MARIE HENRI NAVIER (1785–1836)
Extended equations of motion to include "molecular" forces.

AUGUSTIN LOUIS de CAUCHY (1789–1857)
Contributed to the general field of theoretical hydrodynamics and to the study of wave motion.

GOTTHILF HEINRICH LUDWIG HAGEN (1797–1884)
Conducted original studies of resistance in and transition between laminar and turbulent flow.

JEAN LOUIS POISEUILLE (1799–1869)
Performed meticulous tests on resistance of flow through capillary tubes.

HENRI PHILIBERT GASPARD DARCY (1803–1858)
Performed extensive tests on filtration and pipe resistance; initiated open-channel studies carried out by Bazin.

JULIUS WEISBACH (1806–1871)
Incorporated hydraulics in treatise on engineering mechanics, based on original experiments; noteworthy for flow patterns, nondimensional coefficients, weir, and resistance equations.

TABLE 1.9 (continued)

WILLIAM FROUDE (1810–1879)
Developed many towing-tank techniques, in particular the conversion of wave and boundary layer resistance from model to prototype scale.

ROBERT MANNING (1816–1897)
Proposed several formulas for open-channel resistance.

GEORGE GABRIEL STOKES (1819–1903)
Derived analytically various flow relationships ranging from wave mechanics to viscous resistance—particularly that for the settling of spheres.

ERNST MACH (1838–1916)
One of the pioneers in the field of supersonic aerodynamics.

OSBORNE REYNOLDS (1842–1912)
Described original experiments in many fields—cavitation, river model similarity, pipe resistance—and devised two parameters for viscous flow; adapted equations of motion of a viscous fluid to mean conditions of turbulent flow.

JOHN WILLIAM STRUTT, LORD RAYLEIGH (1842–1919)
Investigated hydrodynamics of bubble collapse, wave motion, jet instability, laminar flow analogies, and dynamic similarity.

VINCENZ STROUHAL (1850–1922)
Investigated the phenomenon of "singing wires."

EDGAR BUCKINGHAM (1867–1940)
Stimulated interest in the United States in the use of dimensional analysis.

MORITZ WEBER (1871–1951)
Emphasized the use of the principles of similitude in fluid flow studies and formulated a capillarity similarity parameter.

LUDWIG PRANDTL (1875–1953) 1904
Introduced concept of the boundary layer B.L.T
and is generally considered to be the father of present-day fluid mechanics.

LEWIS FERRY MOODY (1880–1953)
Provided many innovations in the field of hydraulic machinery. Proposed a method of correlating pipe resistance data which is widely used.

THEODOR VON KÁRMÁN (1881–1963)
One of the recognized leaders of twentieth century fluid mechanics. Provided major contributions to our understanding of surface resistance, turbulence, and wake phenomena.

PAUL RICHARD HEINRICH BLASIUS (1883–1970)
One of Prandtl's students who provided an analytical solution to the boundary layer equations. Also, demonstrated that pipe resistance was related to the Reynolds number.

[a] Adapted from Ref. 2; used by permission of the Iowa Institute of Hydraulic Research, The University of Iowa.

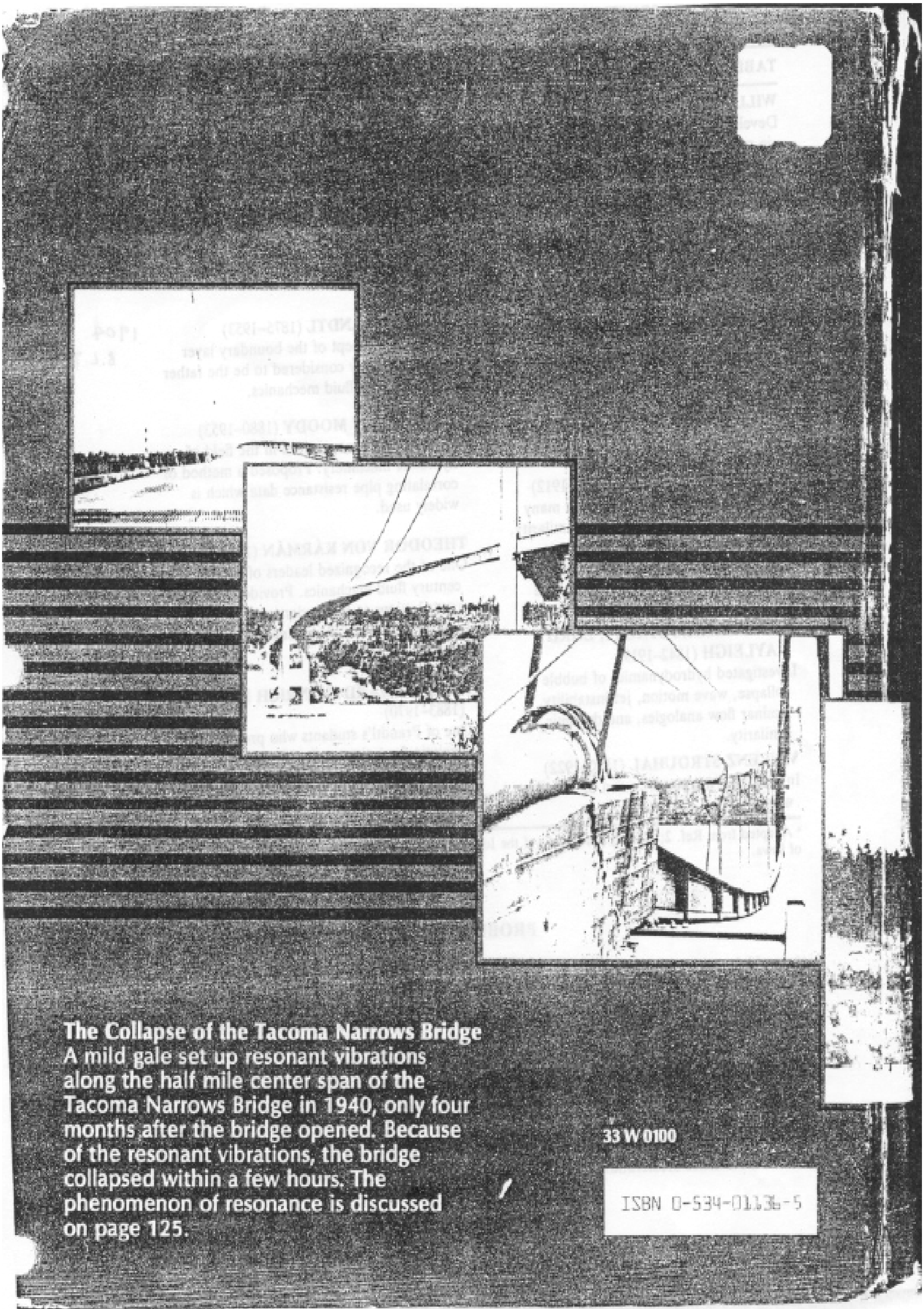
PROBLEMS

Note: Unless specific values of required fluid properties are given in the statement of the problem, use the values found in the tables on the inside of the front cover. Problems designated with an (*) are intended to be solved with the aid of a programmable calculator or a computer.

1.1 Determine the dimensions, in both the *FLT* system and the *MLT* system, for **(a)** the product of force times acceleration, **(b)** the product of force times velocity divided by area, and **(c)** momentum divided by volume.

1.2 Verify the dimensions, in both the *FLT* and *MLT* systems, of the following quantities which appear in Table 1.1: **(a)** angular velocity, **(b)** energy, **(c)** moment of inertia (area), **(d)** power, and **(e)** pressure.

1.3 Verify the dimensions, in both the *FLT* system and the *MLT* system, of the following quantities which appear in Table 1.1: **(a)** fre-


The Collapse of the Tacoma Narrows Bridge
A mild gale set up resonant vibrations along the half mile center span of the Tacoma Narrows Bridge in 1940, only four months after the bridge opened. Because of the resonant vibrations, the bridge collapsed within a few hours. The phenomenon of resonance is discussed on page 125.
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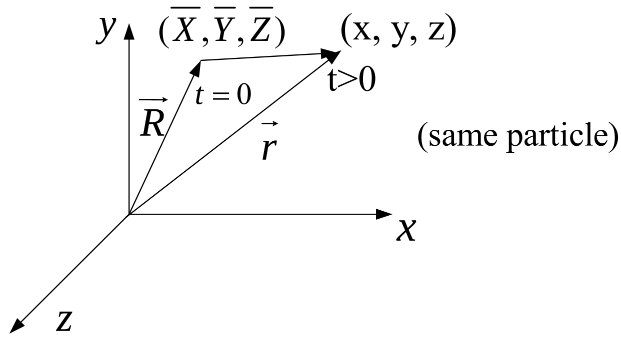
Chapter 2 Derivation of the Equations of motion

2.1 Description of fluid motion

Consider a specific particle

At t=0, x = X, y = Y, z = Z

At t>0,



$$x = X + \int_0^t \left(\frac{dx}{dt}\right) dt$$

$$y = Y + \int_0^t \left(\frac{dy}{dt}\right) dt$$

$$z = Z + \int_0^t \left(\frac{dz}{dt}\right) dt$$

or
$$\vec{r} = \vec{R} + \int_0^t \left(\frac{d\vec{r}}{dt}\right) dt \tag{2.1}$$

$$\vec{r} = \vec{r}(\vec{R}, t)$$

material position vector (become it represents the coordinate, used to “tag” on identify a given particle)

spatial position vector
(become it locate a particle in space)

velocity of a particle = time rate of change of the spatial position vector for this particle.

$$\vec{V} = \left(\frac{d\vec{r}}{dt}\right)_{\vec{R}} \equiv \frac{D\vec{r}}{Dt} \tag{2.2}$$

Where $\frac{D}{Dt}$ denote the time derivation is evaluated with the material coordinate held constant, it is called a material derivative. In this approach, we describe the fluid particle as if we are siding on this fluid particle. The fluid motion is described by material coordinate and time and is often referred to as the Lagrangian description. In general, if $\vec{Q}$ is a property of the fluid, we have

$$Q = Q(\vec{R}, t)$$

That is, we measure the properties $\vec{Q}$ while moving with a particle. The time rate of change of $\vec{Q}$ is

$$\left(\frac{dQ}{dt}\right)_{\vec{k}} = \lim_{\triangle t\to\infty}\left[\frac{Q(t+\triangle t)-Q(t)}{\triangle t}\right]_{\vec{R}} = \frac{DQ}{Dt}$$

Note that $Q(t+\triangle t)$ and Q(t) us the properties of Q for the same fluid particle.

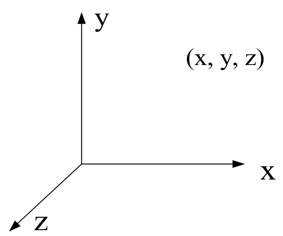
However, Q may be measured at a point fixed in space by a instrument. That is

$$Q = Q(x,y,z,t) = Q(\vec{r},t) \tag{2.3}$$

This is called a 〝<u>Euler Description</u>〞.

If the spatial coordinate $\vec{r}$ are held constant while we take the limit

$$\left(\frac{dQ}{dt}\right)_{\vec{r}} = \frac{\partial Q}{\partial t} = \frac{dy}{dx}\lim_{\triangle t\to\infty}\left[\frac{Q(t+\triangle t)-Q(t)}{\triangle t}\right]_{\vec{r}}$$



The relation between $\left(\frac{dQ}{dt}\right)_{\vec{r}}$ and $\left(\frac{dQ}{dt}\right)_{\vec{R}}$ is as follows:

$$Q = Q(\vec{r},t) = Q(\vec{r}(\vec{R},t),t)$$

$$= Q[x(\vec{R},t), y(\vec{R},t), z(\vec{R},t), t]$$

$$\left(\frac{dQ}{dt}\right)_{\vec{R}} = \frac{DQ}{Dt} = \underbrace{\left(\frac{\partial Q}{\partial x}\right)\left(\frac{dx}{dt}\right)_{\vec{R}}}_{u} + \underbrace{\left(\frac{\partial Q}{\partial y}\right)\left(\frac{dy}{dt}\right)_{\vec{R}}}_{v} + \underbrace{\left(\frac{\partial Q}{\partial z}\right)\left(\frac{dz}{dt}\right)_{\vec{R}}}_{w} + \underbrace{\frac{\partial Q}{\partial t}}_{=\left(\frac{dQ}{dt}\right)_{\vec{r}}}$$

$$\therefore \quad \boxed{\left(\frac{dQ}{dt}\right)_{\vec{R}} = \left(\frac{dQ}{dt}\right)_{\vec{r}} + \vec{V}\bullet\nabla Q} \tag{2.4a}$$

or

$$\boxed{\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + \vec{V}\bullet\nabla Q} \tag{2.4b}$$

(Convective derivative)

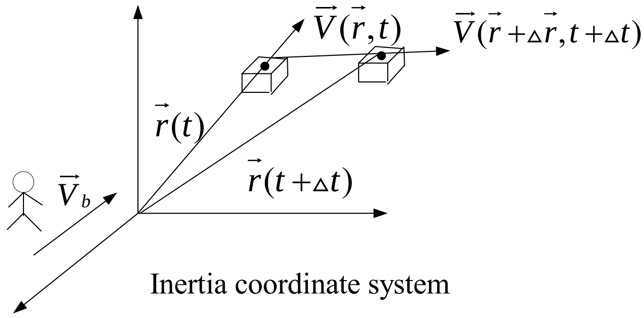
(Unsteady derivative or local derivative)

(Material Derivative substantial Euler derivative)

If moves with the same does stay in a stationary location, nor moves with same velocity as the fluid particle $(\vec{V})$, but moves with velocity $\vec{V_b}$, then

$$Q = Q(\vec{r},t)$$

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial X}\frac{dx}{dt} + \frac{\partial Q}{\partial y}\frac{dy}{dt} + \frac{\partial Q}{\partial z}\frac{dz}{dt}$$



and

$$\boxed{\frac{DQ}{dt}=\frac{\partial Q}{\partial t}+\vec{V}\bullet\nabla Q} \tag{2.5}$$

$$\left(\frac{dQ}{dt}\right)_{\vec{r}}=\frac{\partial Q}{\partial t}$$

$\left(\frac{dQ}{dt}\right)_{observer}$ = The time rate of change of fluid property $Q(\vec{r},t)$ measured by the observer.

$$=\frac{\partial Q}{\partial t}+(\vec{V}-\vec{V}_b)\bullet\nabla Q$$

Similarly:

$\vec{a}$ = acceleration of a fluid particle

= time rate of change of the fluid particle

$$=\left(\frac{d\vec{V}}{dt}\right)_{\vec{R}}=\frac{D\vec{V}}{Dt}=\frac{\partial\vec{V}}{\partial t}+\vec{V}\bullet\nabla\vec{V} \tag{2.6}$$

Note: (1) Observer riding with the fluid particle would describe his acceleration in terms of a single vector $\vec{a}$; the fixed observer would note the $\vec{V}$, $\nabla\vec{V}$, $\frac{\partial\vec{V}}{\partial t}$, and from these quantities be would deduce the acceleration.

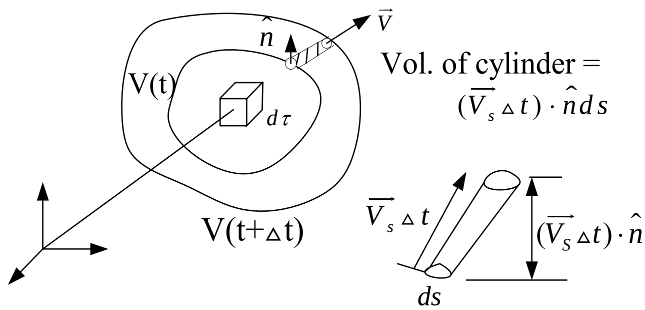
(2) If the flow is steady $\left(\frac{\partial\vec{V}}{\partial t}=0\right)$, the acceleration is not necessarily zero. Since, from (2.6)

$$\vec{a}=\vec{V}\cdot\nabla\vec{V}$$

2.2 Transport Theorem

Consider a volume e.g. a sphere V(t) moving through space so that the velocity of each point of the volume is given by $\vec{V}$. The velocity $\vec{V}$ may be a function of the spatial coordinate. (if the volume is deforming) and time (if the volume is accelerating or decelerating).

$$I(t) = \iiint_{V(t)} Q(\vec{r},t)d\tau$$



$$\frac{dI}{dt} = ?$$

$$\frac{dI}{dt} = \lim_{\triangle t \to 0} \frac{I(t+\triangle t) - I(t)}{\triangle t}$$

$$= \lim_{\triangle t \to 0} \frac{1}{\triangle t}\left[\iiint_{V(t+\triangle t)} Q(\vec{r},t+\triangle t)d\tau - \iiint_{V(t)} Q(\vec{r},t)d\tau\right]$$

($\vec{V}$: fluid velocity as seen by a fixed observer)

<u>Leibnitz's Rule in Calculus</u>:

$$\frac{d}{dt}\int_A^B f(x,t)dt = \int_A^B \frac{\partial f(x,t)}{\partial x}dt + f(x,B)\frac{dB}{dx} - f(x,A)\frac{dA}{dx}$$

where A=A(x), B=B(x) and $A'(x), B'(x)$ are continuous in (a, b),

with $a \le x \le b$ and $A \le t \le B$

$$\because \iiint_{V(t+\triangle t)} Q(\vec{r},t+\triangle t)d\tau = \iiint_{V(t)} Q(\vec{r},t+\triangle t)d\tau + \text{part changing be cause of volume.}$$

$$= \iiint_{V(t)} Q(\vec{r},t+\triangle t)d\tau + \triangle t\iint_{S(t)} Q\vec{V}\bullet\hat{n}ds$$

$$\therefore \frac{dI}{dt} = \lim_{\triangle t \to 0}\frac{1}{\triangle t}\left\{\iiint_{V(t)} [Q(\vec{r},t+\triangle t) - Q(\vec{r},t)]d\tau + \triangle t\iint_{S(t)} Q\vec{V}\bullet\hat{n}ds\right\}$$

By Taylor's expansion

$$Q(\vec{r},t+\triangle t) = Q(\vec{r},t) + \frac{\partial Q}{\partial t}\triangle t + h.o.T$$

$$\therefore \frac{d}{dt}\iiint_{V(t)} Q(\vec{r},t)d\tau = \lim_{\triangle t \to 0}\frac{1}{\triangle t}\left[\iiint_{V(t)} \frac{\partial Q}{\partial t}\triangle t\right]d\tau + \lim_{\triangle t \to 0}\frac{1}{\triangle t}\triangle t\iint_S Q\vec{V}\cdot nds$$

$$+ \lim_{\triangle t \to 0}\frac{1}{\triangle t}\left[\iint \quad \right](\triangle t)^2$$

so $$\boxed{\frac{d}{dt}\iiint_{V(t)} Q(\vec{r},t)d\tau = \iiint_{V(t)} \frac{\partial Q}{\partial t}d\tau + \oiint_{s(t)} Q\vec{V}\cdot\hat{n}ds} \tag{2.7}$$

″ General Transport Theorem, 3-D Leibnitz's Rule″

Special Cause:

(1) If the volume is fixed in space. ($\vec{V}$ =0 on the S(t), V(t)= fixed ≡V)

$$\boxed{\frac{d}{dt}\iiint_V Qd\tau = \iiint_V \frac{\partial Q}{\partial t}d\tau} \tag{2.8}$$

(2) If the mass is fixed. (closed system, $\frac{d}{dt}=\frac{D}{Dt}$)

$$\boxed{\frac{D}{Dt}\iiint_{V(t)} Qd\tau = \iiint_{V(t)} \frac{\partial Q}{\partial t}d\tau + \oiint_{S(t)} Q\vec{V}\cdot\hat{n}ds} \tag{2.9}$$

″ Reynolds Transport Theorem ″

By Divergence Theorem

$$\iiint_V \nabla\cdot\vec{A}d\tau = \oiint \vec{A}\cdot\hat{n}ds$$

We obtain

$$\frac{D}{Dt}\iiint_{V(t)} Qd\tau = \iiint_{V(t)} [\frac{\partial Q}{\partial t} + \nabla\cdot(\vec{V}Q)]d\tau$$

As V(t)→0

$$\frac{D}{Dt}[Q\Delta\tau] = [\frac{\partial Q}{\partial t} + \nabla\cdot(\vec{V}Q)]\Delta\tau$$

As Q=1

$$\frac{1}{\Delta\tau}\frac{D(\Delta\tau)}{Dt} = \nabla\cdot\vec{V}$$

take limit

$$\underbrace{\lim_{\Delta\tau\to 0}\frac{1}{\Delta\tau}\frac{D(\Delta\tau)}{Dt}}_{\text{Rate of the volume change = dilatation}} = \nabla\cdot\vec{V}$$

Therefore:

if $\boxed{\nabla\cdot\vec{V}=0}$ ↔ volume strain is zero (2.10)

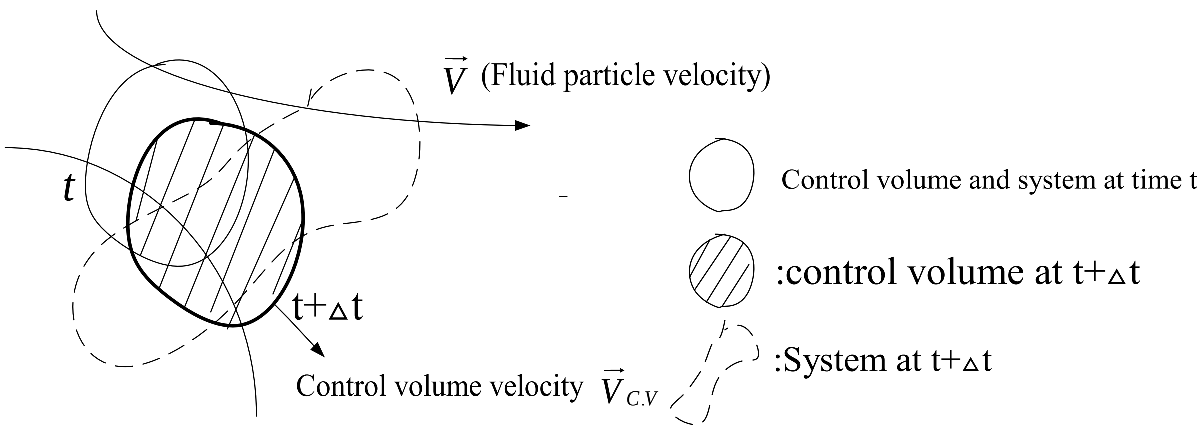
↔ incompressible

This is the basic definition of ″ incompressible″ .

Supplementary material

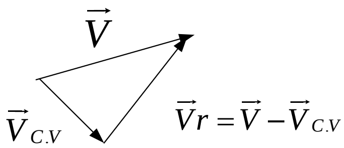
$$\begin{cases} \vec{V}: & \text{Fluid velocity seen by a fixed observer.} \\ \vec{V}_{C.V}: & \text{c.v velocity seen by a fixed observer.} \\ \vec{V}_{C.S}: & \text{c.s velocity seen by a fixed observer.} \\ \vec{V}r: & \text{Fluid velocity seen by a fixed observer moving with the c.s.} \end{cases}$$

$\rightarrow$ **(see note 5-1 back)**



If the absolution fluid velocity is $\vec{V}$, then the fluid velocity relative to moving control surface $\vec{V}r$ is

$$\boxed{\vec{V}r = \vec{V} - \vec{V}_{C.V}} \qquad (4.7)$$



That is, $\vec{V}r$ is the velocity of the flow as seen by an observer moving with velocity $\vec{V}_{C.V}$. For this observer, the control volume is fixed, this E.g. (4.5)or(4.6) can be applied if $\vec{V}$ is replaced by $\vec{V}r$, that is

$$\boxed{\frac{D}{Dt}\int_{sys}\rho b dV = \frac{\partial}{\partial t}\int_{cv}\rho b dV + \int_{cs}\rho b \vec{V}_r \cdot \hat{n} dA} \qquad (4.8)$$

Where $\vec{V}r$ is given in E.g. (4.7)

(3) If the control volume is moving with $\vec{V}_{C.V}$ and the volume is deforming. Then the volume of the control surface $\vec{V}_{C.S}$ will not be the same as $\vec{V}_{C.V}$, we then have Reynold Transport Theorem as (4.8) except that

$$\boxed{\vec{V}r = \vec{V} - \vec{V}_{C.S}} \qquad (4.9)$$

2.3 Conservation of Mass

(1) For a closed system: ($\dot{m} = 0$, Lagrangian Description)

$$\boxed{\frac{D}{Dt}\iiint_{V(t)} \rho d\tau = 0}$$

(2.9)

$$= \iiint_{V(t)} \frac{\partial \rho}{\partial t} d\tau + \oiint_{s(t)} \rho \vec{V} \cdot \hat{n} ds$$

$$= \iiint_{V(t)} [\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V})] d\tau$$

if V(t) is arbitrary and the integrand is continuous, then

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0} \qquad \text{" Continuity equation"} \qquad (2.11)$$

(Since it is continuous in the 1st order)

or

$$\underbrace{\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho}_{= (\frac{d\rho}{dt})_{\vec{R}} = \frac{D\rho}{Dt}} + \rho \nabla \cdot \vec{V} = 0$$

$$\Rightarrow \qquad \boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0} \qquad (2.12)$$

Special Cases:

(a) For a steady flow: ($\frac{\partial}{\partial t} = 0$)

$$\Rightarrow \quad \nabla \cdot (\rho \vec{V}) = 0$$

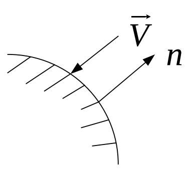
(b) For a incompressible flow: ($\nabla \cdot \vec{V} = 0$)

E.g. (2.12) $\Rightarrow \quad \frac{D\rho}{Dt} = 0$ (This implies that ρ is constant along a streamline. ρ is not a constant everywhere, but $\rho = \rho(\vec{x}, t)$ in general.)

(2) For a fixed region:

$$\begin{bmatrix} \text{Time rate of increase of} \\ \text{mass within the C.V} \end{bmatrix} = \begin{bmatrix} \text{Not influx of mass across the} \\ \text{control surface} \end{bmatrix}$$

$$\frac{d}{dt}\iiint_{V.\,fluid} \rho d\tau = -\oiint_{S.\,fixed} \rho \vec{V} \cdot \hat{n} ds$$



Since V, S is fixed, from E.g. (2.8) with Q= ρ , we have

$$\frac{d}{dt}\iiint_V \rho d\tau = \iiint_V \frac{\partial \rho}{\partial t} d\tau$$

Therefore

$$\boxed{\iiint_{\underset{\uparrow\ \text{fixed}}{V}} \rho d\tau = -\oiint_{\underset{\uparrow\ \text{fixed}}{S}} \rho \vec{V} \cdot \hat{n} ds} \quad \text{" conservation of mass"} \qquad (2.13)$$

Supplementary material

$$\begin{cases} \vec{V}: & \text{Velocity of fluid particle seen by a fixed observer.} \\ \vec{V}_{C.V}: & \text{Velocity of control volume seen by a fixed observer.} \\ \vec{V}_{C.S}: & \text{Velocity of control surface seen by a fixed observer.} \\ \vec{V}r: & \text{Velocity of fluid particle seen by a observer moving with the control volume.} \end{cases}$$

① For non-deforming, no-moving control volume

$$\begin{cases} \vec{V}_{C.V}=0, \quad \vec{V}r=\vec{V} \\ \vec{V}_{C.S}=0 \end{cases}$$

② For non-deforming, moving control volume

$$\begin{cases} \vec{V}_{C.V}=\vec{V}_{C.S} \\ \vec{V}= \vec{V}r+\vec{V}_{C.S} \qquad \text{or } \vec{V}r= \vec{V} - \vec{V}_{C.V} = \vec{V} - \vec{V}_{C.S} \end{cases}$$
$$= \vec{V}r+\vec{V}_{C.V}$$

③ For deforming, moving control volume

$$\begin{cases} \vec{V}_{C.V} \neq \vec{V}_{C.S} \\ \vec{V}r=\vec{V}-\vec{V}_{C.S} \qquad \text{but } \boxed{\vec{V}r \neq \vec{V}-\vec{V}_{C.V}} \end{cases}$$

If the C.V is non-deformed and moving with a velocity of $\vec{V}_{C.V}$, then we have derive in chapter 4 that

$$\boxed{\vec{V}=\vec{V}r+\vec{V}_{C.V}} \tag{5.5}$$

Where $\vec{V}$ is the absolute velocity of the fluid seen by a stationary observer in a fixed coordinate system, and $\vec{V}r$ is the fluid velocity seen by an observer moving with the control volume. The control volume expression of the continuity equation is

$$\boxed{\frac{\partial}{\partial t}\int_{C.V}\rho dV+\int_{C.S}\rho \vec{V}_r \cdot \hat{n} dA=0} \tag{5.6}$$

If the control volume is deforming and moving, then the velocity of the surface $\vec{V}_{C.S}$ and the velocity of the control volume $\vec{V}_{C.V}$ as seen by a fixed observer in a stationary coordinate. System will not be the same. The relation between $\vec{V}$ (absolution fluid velocity.) and $\vec{V}r$ (relative velocity referenced to the control surface.) is

$$\boxed{\vec{V}=\vec{V}_r+\vec{V}_{C.S}} \tag{5.7}$$

and the control volume, expression of the continuity equation is remained the same as equation. (5.6)

2.4 Equation of Change for momentum

Newton's second low

$$\vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt}$$

applies only for a point particle of fixed mass m. For a closed system (Lagrangian description), it become

$$\frac{D}{Dt}\iiint_{V(t)} \rho\vec{V}d\tau = \sum\vec{F} \qquad (2.14)$$

The external not forces include forces acting on the body (volume) and on the surface, namely.

$$\sum\vec{F} = \vec{F}_{body} + \vec{F}_{surface}$$

Neglecting magnetic & electrical effect, the only body force is due to the gravitational force, thus

$$\vec{F}_{body} = \iiint_{V(t)} \rho\vec{f}d\tau$$

Where $\vec{f}$ represent the body force per unit mass.

For any arbitrary position, the surface stresses (surface force/area) not only depend on the direction of the force, but also on the orientation of the surface. Therefore, the surface stress is a second order tension, and is denoted by $\overleftrightarrow{\sigma}$.

Before we involve on the derivation of $\vec{F}_{surface}$, we need to know more about tension.

″ pressure″ means the normal force per unit area acted on the fluid particle>

As the fluid is static, the pressure of the fluid is called hydrostatic pressure. Since the fluid is motionless, the fluid is in equilibrium, therefore the

(Hydrostatic pressure = thermodynamic pressure)

As the fluid is in motion, the 3 principal normal stresses are not necessary equal, and the fluid is not in equilibrium. Therefore, the hydrodynamic pressure is defined by

$$\textbf{(Hydrostatic pressure)} \equiv \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

and which is not equal to the thermodynamic pressure either. Later we will prove that

$$\textbf{(Hydrostatic pressure)} = \underbrace{\textbf{thermodynamic pressure}}_{=\text{(Hydrostatic pressure)}} + \frac{1}{3}\lambda'$$

Supplementary material

5.2 Conservation of momentum

Consider a particular moment when the control volume is coincide with the control volume, then

$$\sum \vec{F}_{system} = \sum \vec{F}_{C.V} \qquad (5.7)$$

Newton's 2nd law for the control mass system is

$$\underbrace{\frac{D}{Dt}\int_{sys} \rho \vec{V} dV}_{\text{= time rate of change of the linear momentum of the system}} = \underbrace{\sum \vec{F}_{sys}}_{\text{= Sum of external forces acting on the system}} \qquad (5.8)$$

From Reynolds Transport Theorem for a fixed spaced, non-deforming control volume.

$$\frac{D}{Dt}\int_{sys} \rho \vec{V} dV = \frac{\partial}{\partial t}\int_{C.V} \vec{V} \rho dV + \int_{C.S} \vec{V} \rho \vec{V} \cdot \hat{n} dA$$

Apply (5.7) & (5.8) into above equation, we can get the momentum equation for a control volume.

$$\boxed{\underbrace{\frac{\partial}{\partial t}\int_{C.V} \vec{V} \rho dV}_{\text{= time rate of change of the linear momentum of the system}} + \underbrace{\int_{C.S} \vec{V} \rho \vec{V} \cdot \hat{n} dA}_{\text{= not rate of flow of linear momentum through the C.S.}} = \sum \vec{F}_{C.V}} \qquad (5.9)$$

Remark:

If the control volume is non-deforming, but moves with a velocity of $\vec{V}_{C.V}$, then we may take b= $\vec{V}$ in equation (4.8), and get

$$\frac{D}{Dt}\int_{sys} \vec{V} \rho dV = \frac{\partial}{\partial t}\int_{C.V} \vec{V} \rho dV + \int_{C.S} \vec{V} \rho \vec{V}_r \cdot \hat{n} dA$$

Combined with (5.7) & (5.8), the above equation can be written as

$$\boxed{\frac{\partial}{\partial t}\int_{C.V} \vec{V} \rho dV + \int_{C.S} \vec{V} \rho \vec{V}_r \cdot \hat{n} dA = \sum \vec{F}_{C.V}} \qquad (5.10)$$

Since $\vec{V} = \vec{V}r + \vec{V}_{C.V}$

Equation (5.10) $\Rightarrow$

$$\boxed{\frac{\partial}{\partial t}\int_{C.V}(\vec{V}_r + \vec{V}_{C.V})\rho dV + \int_{C.S}(\vec{V}_r + \vec{V}_{C.V})\rho\vec{V}_r \cdot \hat{n}dA = \sum \vec{F}_{C.V}} \qquad (5.11)$$

(Non-deforming + moving C.V)

If the flow is steady, then

$$\frac{\partial}{\partial t}\int_{C.V}(\vec{V}_r + \vec{V}_{C.V})\rho dV = 0$$

and from the continuity equation

$$\frac{\partial}{\partial t}\int_{C.V}\rho dV + \int_{C.S}\rho\vec{V}_r \cdot \hat{n}dA = 0 \qquad (*)$$

The momentum equation of (5.11) reduces to

$$\int_{C.S}\vec{V}_r\rho\vec{V}_r \cdot \hat{n}dA + \underbrace{\int_{C.S}\vec{V}_{C.V}\rho\vec{V}_r \cdot \hat{n}dA}_{=\vec{V}_{C.V}\int_{C.S}\rho\vec{V}_r \cdot \hat{n}dA = 0 \text{ from equation}(*)} = \sum \vec{F}_{C.V}$$

or

$$\boxed{\int_{C.S}\vec{V}_r\rho\vec{V}_r \cdot \hat{n}dA = \sum \vec{F}_{C.V}} \qquad (5.12)$$

(For a non-deforming C.V moving with a constant velocity in <u>a steady state flow</u>)

Aside: A second order tension, called a dyad and denoted as $\vec{A}\vec{B}$, satisfies the following properties:

$$(\vec{A}\vec{B})\cdot\vec{C}=\vec{A}(\vec{B}\cdot\vec{C})$$

$$\vec{C}\cdot(\vec{A}\vec{B})=(\vec{C}\cdot\vec{A})\vec{B}$$

A unit tension, $\overleftrightarrow{II}$, is a tension with

$$\overleftrightarrow{\text{II}}\cdot\vec{C}=\vec{C},\quad \vec{C}\cdot\overleftrightarrow{\text{II}}=\vec{C}$$

In a Cartesian coordinate system,

$$\overleftrightarrow{\text{II}}=\vec{i}\vec{i}+\vec{j}\vec{j}+\vec{k}\vec{k}$$

Now, back to the issue of surface forces, as the fluid is in static equilibrium, the only stress is the normal stresses, thus

$$\overleftrightarrow{\sigma}=-\underbrace{p}_{=\text{ (hydrostatic pressure)}}\overleftrightarrow{\text{II}} \qquad \left[\overleftrightarrow{\text{II}}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right]$$

If the fluid is in motion, we assume:

$$\overleftrightarrow{\sigma}=\underbrace{-p\overleftrightarrow{\text{II}}}_{\text{Thermodynamic pressure}}+\underbrace{\overleftrightarrow{\tau}}_{\text{Viscous stress}} \tag{2.15}$$

Question:

Are ″hydrostatic pressure″, ″hydrodynamic pressure″ and ″thermodynamic pressure″ the same? We will answer this question later.

The surface forces thus become

$$\vec{F}=\oiint_{s(t)}(\hat{n}\cdot\overleftrightarrow{\sigma})ds$$

$$=\oiint_{s(t)}-p\hat{n}ds+\oiint_{s(t)}(\overleftrightarrow{\tau}\cdot\hat{n})ds$$

$$=\iiint_{R(t)}[-\nabla p+\nabla\cdot\overleftrightarrow{\tau}]d\tau$$

Equation (2.14) thus become

$$\underbrace{\frac{D}{Dt}\iiint_{R(t)}\rho\vec{V}d\tau}_{} = \iiint_{R(t)}\rho\vec{f}d\tau + \iiint_{R(t)}[-\nabla p + \nabla\cdot\overleftrightarrow{\tau}]d\tau$$

$$\overset{(2.9)}{=} \iiint_{R(t)}\frac{\partial(\rho\vec{V})}{\partial t}d\tau + \oiint_{S(t)}(\rho\vec{V})\vec{V}\cdot\hat{n}ds$$

$$= \iiint_{R(t)}[\frac{\partial(\rho\vec{V})}{\partial t} + \nabla\cdot(\rho\vec{V}\vec{V})]d\tau$$

$$\Rightarrow \qquad \boxed{\frac{\partial(\rho\vec{V})}{\partial t} + \nabla\cdot(\underbrace{\rho\vec{V}\vec{V}}_{\text{momentum flux tensor}}) = \rho\vec{f} - \nabla p + \nabla\cdot\overleftrightarrow{\tau}} \qquad (2.16)$$

or

$$\frac{D}{Dt}\iiint_{R(t)}\rho\vec{V}d\tau = \frac{D}{Dt}\iiint_{R(t)}\vec{V}dm = \iiint_{R(t)}\frac{D\vec{V}}{Dt}dm = \iiint_{R(t)}\rho\frac{D\vec{V}}{Dt}d\tau$$

$\because \frac{D}{Dt}$ means we follow a fluid particle, thus the mass dm is a constant, and not a function of time & location.

Equation (2.16) thus has another form of

$$\boxed{\rho\frac{D\vec{V}}{Dt} = \rho\vec{f} - \nabla p + \nabla\cdot\overleftrightarrow{\tau}} \qquad (2.17)$$

Equation (2.17) may be derived from (2.16) either

$$\underset{(2.16)}{\text{L.H.S}} = \frac{\partial(\rho\vec{V})}{\partial t} + \nabla\cdot(\rho\vec{V}\vec{V})$$

$$= \rho\frac{\partial\vec{V}}{\partial t} + \frac{\partial\rho}{\partial t}\vec{V} + \vec{V}\nabla\cdot(\rho\vec{V}) + (\rho\vec{V})(\nabla\cdot\vec{V})$$

$$= \rho\left[\frac{\partial\vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V}\right] + \vec{V}\underbrace{\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V})\right]}_{(=0\text{ from continuity equation})}$$

$$= \rho\frac{D\vec{V}}{Dt}$$

By the tenor operation, we show that the left-hand side of the equation (2.16) & (2.17) are identical.

Chapter 3 Exact Solution of N-S Equation

Assumptions: ① Constant Density (Incompressible Flow)

② Constant μ, k, $C_v, C_p \quad e = C_v T$

③ No body forces

3.1 Parallel Flow

$$\vec{V} = u\hat{i} + \underbrace{v}_{=0}\hat{j} + \underbrace{w}_{=0}\hat{k}$$

or

$v = w = 0$, but

$u = u(x, y, z, t)$, $\quad p = p(x, y, z, t)$, $\quad T = T(x, y, z, t)$

1) Continuity equation:

$$\nabla \cdot \vec{V} = 0 \;\Rightarrow\; \frac{\partial u}{\partial x} + \cancelto{0}{\frac{\partial v}{\partial y}} + \cancelto{0}{\frac{\partial w}{\partial z}} = 0$$

$\Rightarrow \dfrac{\partial u}{\partial x} = 0 \;\Rightarrow\;$ u does not depend on x

or $\quad u = u(y, z, t)$

2) Momentum equation:

$$\rho\left[\frac{\partial V}{\partial t} + \vec{V} \cdot \nabla \vec{V}\right] = -\nabla P + \mu \nabla^2 \vec{V}$$

or

$$\begin{cases} \rho\left[\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z}\right] = -\dfrac{\partial p}{\partial x} + \mu\left[\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right] \\ \rho\left[\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z}\right] = -\dfrac{\partial p}{\partial y} + \mu\left[\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2}\right] \\ \rho\left[\dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z}\right] = -\dfrac{\partial p}{\partial z} + \mu\left[\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}\right] \end{cases}$$

$$\Rightarrow \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0 \;\Rightarrow\; p = p(x, t)$$

$$\boxed{\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)} \tag{3.1}$$

if we $\quad \frac{\partial}{\partial x}\left[\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\right]$

$$\Rightarrow\ \rho\frac{\partial}{\partial t}(\cancelto{0}{\frac{\partial u}{\partial x}})=\frac{\partial}{\partial x}(-\frac{\partial p}{\partial x})+\mu\left[\frac{\partial^2}{\partial y^2}(\cancelto{0}{\frac{\partial u}{\partial x}})+\frac{\partial^2}{\partial z^2}(\cancelto{0}{\frac{\partial u}{\partial x}})\right]$$

$$\Rightarrow\ \frac{\partial^2 p}{\partial x^2}=0\ \text{ or }\ \frac{\partial p}{\partial x}=\ \text{function of t only.}$$

$$\therefore\quad u=u(t,y,z)\,,\quad p=p(x,t)\,,\quad \frac{\partial p}{\partial x}=f_n(t)$$

3) Energy equation:

Eq. (2.40) $\Rightarrow$

$$\rho C_v\left[\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+\cancel{v}\frac{\partial T}{\partial y}+\cancel{w}\frac{\partial T}{\partial z}\right]=2\mu\left\{(\cancel{\frac{\partial u}{\partial x}})^2+(\cancelto{(v=0)}{\frac{\partial v}{\partial y}})^2+(\cancelto{(w=0)}{\frac{\partial w}{\partial z}})^2\right.$$

$$\left.+\frac{1}{2}\left[(\frac{\partial u}{\partial y}+\cancel{\frac{\partial v}{\partial x}})^2+(\frac{\partial u}{\partial z}+\cancel{\frac{\partial w}{\partial x}})^2+(\frac{\partial v}{\partial z}+\cancel{\frac{\partial w}{\partial y}})^2\right]\right\}+k(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2})$$

$$\Rightarrow\boxed{\rho C_v(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x})=\mu\left[(\frac{\partial u}{\partial y})^2+(\frac{\partial u}{\partial z})^2\right]+k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)} \tag{3.2}$$

$$\therefore\quad T=T(t,x,y,z)$$

3.1.1 Steady, Parallel, 2-D Flow

$$(\frac{\partial}{\partial t}=0\,,\ \frac{\partial}{\partial z}=0)$$

From the pressure discussion, we know

$$u=u(y)\,,\quad p=p(x)\,,\quad T=T(x,y)\,,\quad \frac{\partial p}{\partial x}=\frac{dp}{dx}=\text{constant}$$

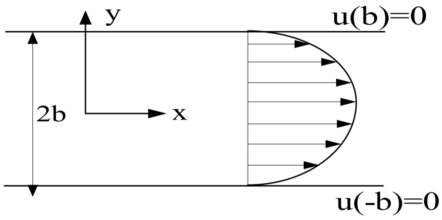
The Equation of motion become

$$\left\{\begin{aligned}&\mu\frac{\partial^2 u}{\partial y^2}=\frac{dp}{dx}=\text{constant} &\quad(3.3a)\\[2ex] &\rho C_v u\frac{\partial T}{\partial x}=\mu(\frac{\partial u}{\partial y})^2+k\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right] &\quad(3.3b)\end{aligned}\right.$$

integrate Eq (3.3a), we have

$$u(y) = \frac{y^2}{2\mu}(\frac{dp}{dx}) + C_1 y + C_2 \tag{3.4}$$

a) **Poiseuille (pressure-deriver) duct flows**:



u(b)=u(-b)=0

Eq. (3.4) $\Rightarrow$

$$u(y) = \frac{1}{2\mu}(\frac{dp}{dx})(b^2 - y^2) \qquad \text{parabolic profile}$$

The shear stress is

$$\tau_{ij} = 2\mu\varepsilon_{ij} = \mu(\frac{\partial V_j}{\partial x_i} + \frac{\partial V_i}{\partial x_j})$$

$$\tau_{11} = \tau_{xx} = \mu(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}) = 0 \quad (\because \frac{\partial u}{\partial x} = 0 \quad \text{from continuity equation})$$

$\Rightarrow$ No normal shearing stresses

$$\tau_{12} = \tau_{21} = \mu(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) = \mu\frac{du}{dy}$$

$$\therefore \quad \tau = \mu\frac{du}{dy} = \frac{dp}{dx}y$$

Thus the wall function is ($\tau_w = \left|\tau_{y=\pm b}\right|$)

$$\boxed{\tau_w = \frac{dp}{dx}b}$$

From the energy equation:

$$\rho C_v u\frac{\partial T}{\partial x} = \mu(\frac{dp}{dx})^2 y^2 + k(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2})$$

If the channel is infinitely long, we may assume that the temperature distribution is fully-developed, i.e.

$$\frac{\partial T}{\partial x} = 0 \quad \text{or} \quad T = T(y) \quad \text{only}$$

Energy equation become

$$k\frac{d^2T}{dy^2} = -\frac{1}{\mu}(\frac{dp}{dx})^2 y^2$$

integrate twice

$$T(y) = -\frac{1}{\mu k}\left(\frac{dp}{dx}\right)^2 \frac{y^4}{12} + C_3 y + C_4$$

If the B.C'S are: $T(b) = T_w^+$, $T(-b) = T_w^-$, then

$$\boxed{T(y) = \frac{T_w^+ + T_w^-}{2} + \frac{T_w^+ + T_w^-}{2}\frac{y}{b} + \frac{b^4}{12k\mu}\left(\frac{dp}{dx}\right)^2\left(1 - \frac{y^4}{b^4}\right)}$$

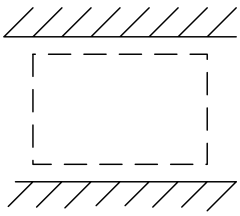
When we calculate u(y), we are actually interested in the value of τ_w. Similarly, as we solve the temperature distribution, we want to know the heat transfer on the walls.

<u>Aside</u>:

In the temperature section, we mentioned that

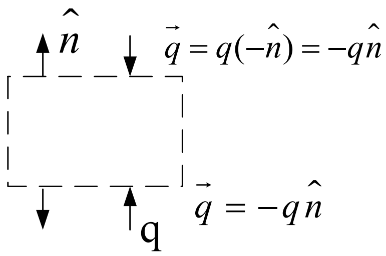
$$\left(k\frac{\partial T}{\partial n}\right)_{fluid} = q_{solid \to fluid}$$

For the current case



On upper surface

On lower surface



therefore $\vec{q} = -k\nabla T$

$$-q\hat{e}_n = -k\left.\frac{\partial T}{\partial n}\right|_{fluid} \hat{e}_n \Rightarrow \boxed{q_{s\to F} = k\left.\frac{\partial T}{\partial n}\right|_{fluid}}$$

However, this is not a good way become $\hat{e}_n$ always change its direction for a fixed coordinate frame. Therefore, we may take the positive valve of q as heat transfer in the direction of the positive-coordinate axis, then

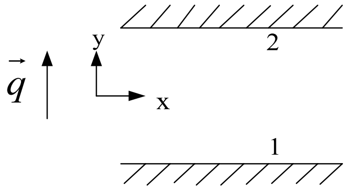
$$\vec{q} = -k\nabla p$$

$$\Rightarrow \quad q_x\vec{i} + q_y\vec{j} + q_z\vec{k} = -k\left(\frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j} + \frac{\partial T}{\partial z}\vec{k}\right)$$

$$\Rightarrow \quad q_x = -k\frac{\partial T}{\partial x},\ q_y = -k\frac{\partial T}{\partial y},\ q_z = -k\frac{\partial T}{\partial z}$$

At any point, if $q_x > 0$, it means the x-component of the heat transfer at this point is in the +x-axis direction.

For this case:



Therefore, we set $\vec{q}$ in the direction of +y, then

$$q = -k\frac{dT}{dy}$$

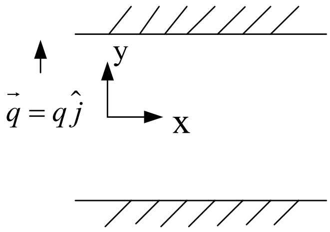
(i) If $q_{\text{at point①}} > 0$, it means q is transferred upward, therefore, it is from the lower wall to the fluid.

(ii) If $q_{\text{at point①}} < 0$, the heat is transferred downward, therefore, it is from the fluid to the lower wall.

(iii)If $q_{\text{at point②}} > 0 \Rightarrow$ fluid to upper wall.

(iv)If $q_{\text{at point②}} < 0 \Rightarrow$ upper wall to the fluid #

Take



q>0, heat transfer upward

q<0, heat transfer downward

then $$q = -k\frac{dT}{dy}$$

or $$q = -k\left[\frac{T_w^+ + T_w^-}{2}\frac{1}{b} - \frac{b^3}{3\mu k}\left(\frac{dp}{dx}\right)^2\left(\frac{y}{b}\right)^3\right]$$

Hence

$$q(b) = -k\left[\frac{T_w^+ - T_w^-}{2}\frac{1}{b} - \frac{b^3}{3\mu k}\left(\frac{dp}{dx}\right)^2\right] \equiv q^+$$

$$q(-b) = -k\left[\frac{T_w^+ - T_w^-}{2}\frac{1}{b} - \frac{b^3}{3\mu k}\left(\frac{dp}{dx}\right)^2\right] \equiv q^-$$

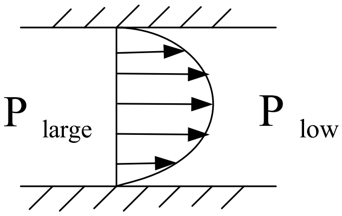
<u>Remark</u>:

(1) $u(y) = -\dfrac{1}{2\mu}(\dfrac{dp}{dx})(b^2 - y^2)$

if $\dfrac{dp}{dx} = 0$, no fluid motion.

$\dfrac{dp}{dx} < 0 \;\Rightarrow\; u(y) = 0$, or the fluid is moved to the right.

Therefore



∵ P is a driving force of the motion

(2) $\tau_w = \pm\dfrac{dp}{dx}b$, why $(\tau_w)_{lower} = -(\dfrac{dp}{dx})b$, while $(\tau_w)_{upper} = (\dfrac{dp}{dx})b$?

since $\vec{\tau}_n = \hat{n} \cdot \overleftrightarrow{\tau}$

$$\vec{\tau}_{\substack{lower\ surface\\ of\ fluid}} = -\hat{j} \cdot (\tau_{mn}\hat{e}_m\hat{e}_n) = -\tau_{jn}\hat{e}_n \qquad \text{(j= 2, n=1, 2, 3)}$$

$$= -\tau_{21}\hat{i} + \tau_{22}\hat{j} + \tau_{23}\hat{k}$$

$$0 \qquad \left[= \mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) = 0\right]$$

(Normal stress) $\quad (\dfrac{\partial}{\partial z} = 0) \quad$ (w=0)

2-D $\quad$ parallel flow

$$= -\tau_w\hat{i} = -(\frac{dp}{dx})b\hat{i} > 0 \qquad (\because (\frac{dp}{dx}) < 0)$$

Therefore $(\tau_w)_{low} < 0$ means $\vec{\tau}_{\text{lower wall}}$ is acted on the negative direction of $\hat{i}$.

Similarly: $\vec{\tau}_{upper\ wall} = +\hat{j} \cdot (\tau_{mn}\hat{e}_m\hat{e}_n) = +\tau_{jn}\hat{e}_n \quad$ (j=2, n=1,2,3)

$$= +\tau_{21}\hat{i} = +(\tau_w)_{upper}\hat{i} = +(\frac{dp}{dx})b\hat{i} < 0$$

Therefore, $\vec{\tau}_{upper\ wall}$ is still in the direction of –x axis.

(3) $q(b) \neq q(-b)$ because $T_w^+ \neq T_w^-$

However, if $T_w^+ = T_w^-$, we know the results that

$$\begin{cases} q^+ = +\dfrac{b^3}{3\mu}(\dfrac{dp}{dx})^2 \\ q^- = -\dfrac{b^3}{3\mu}(\dfrac{dp}{dx})^2 \end{cases}$$

Why these is a difference in sign? Does it mean that one wall is received heat while the other given away the heat? The answer is that

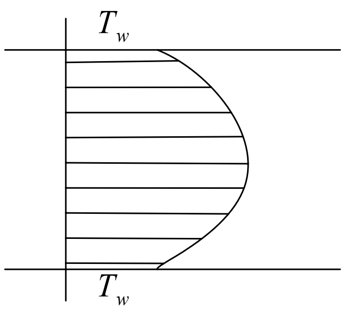
$q^+ > 0 \quad \Rightarrow$ heat transfer upward $\quad \Rightarrow$ from fluid to the upper wall

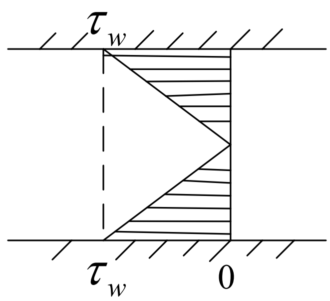
$q^- < 0 \quad \Rightarrow$ heat transfer downward $\Rightarrow$ from fluid to the lower wall

To understand the flow in more detail, let's see the temperature profile for the case of:

$$T_w^+ = T_w^- = T_w$$

$$T(y) = T_w + \underbrace{\frac{b^4}{12k\mu}(\frac{dp}{dx})^2(1-\frac{y^4}{b^4})}_{\geq 0}$$





The shearing stress is

$$\tau = \mu\frac{du}{dy} = \frac{dp}{dx}y$$

Question:

Why the temperature is highest but the shearing stress is minimum (zero) along the centerline?

Answer:

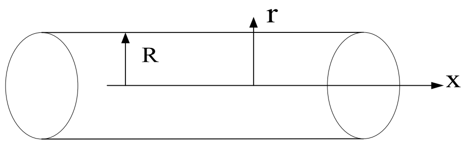
The high viscous force along the walls will produce a large amount of dissipation energy. In turn, it will increase the internal energy of the fluid near the wall. Partial internal energy transport to the wall due to dissipation gradient, $\left[|q_w| = \frac{b^3}{3\mu}(\frac{dp}{dx})^2 \right]$, the rest of viscosity. Along the centerline, the fluid received the diffused energy from upper & lower surface, thus it has the max temperature.

b) Poiseuille (pressure-driven) pipe flow:

(Parallel flow: planar (2D) flow, or Axisymmetric flow.)

In cylindrical coordinate:

$$\vec{V} = u\hat{e}_x + v\hat{e}_r + w\hat{e}_\phi$$



and

$$u = u(r,x),\ \text{v = w =0 (parallel)},\ \frac{\partial}{\partial \phi} = 0 \quad \text{(2-D)}$$

$$\left. \begin{aligned} P &= P(r,x) \\ T &= T(r,x) \end{aligned} \right\} \quad \text{(may be! Write down in this way temperature)}$$

Continuity:

$$\nabla \cdot \vec{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial r}(h_2 h_3 v) + \frac{\partial}{\partial \phi}(h_1 h_3 w) + \frac{\alpha}{\alpha x}(h_1 h_2 u) \right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial x}(ru) \right] = 0$$

$$u \neq f_n(x) \quad \rightarrow \quad \therefore\ \text{u = u(r)}$$

$$\begin{cases} h_1 = h_3 = 1 \\ h_2 = r \\ x_1 = r \\ x_2 = \phi \\ x_3 = X \end{cases}$$

Momentum:

$$\rho\left[\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \right] = -\nabla \mathrm{P} + \mu \nabla^2 \vec{V}$$

$$\nabla = \frac{1}{h_1}\frac{\partial}{\partial x_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial}{\partial x_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial}{\partial x_3}\hat{e}_3$$

$$= \frac{\partial}{\partial x_r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial x_\phi}\hat{e}_\phi + \frac{\partial}{\partial X}\hat{e}_X$$

$$\nabla\vec{V} = \nabla(u\hat{e}_x) = (\nabla u)\hat{e}_x + u\nabla(\cancel{\hat{e}_x}) = (\frac{du}{dr}\hat{e}_r)\hat{e}_x = \frac{du}{dr}\hat{e}_r\hat{e}_x$$

0 (∵ $\hat{e}_x$ is fixed, however $\hat{e}_r, \hat{e}_\phi$ are not)

$$\vec{V}\cdot\nabla\vec{V} = (u\hat{e}_x)\cdot(\frac{du}{dr}\hat{e}_r\hat{e}_x) = (u\frac{du}{dr})(\hat{e}_x\cdot\hat{e}_r)\hat{e}_x = 0$$

$$\nabla p = \frac{\partial p}{\partial r}\hat{e}_r + \frac{\partial p}{\partial x}\hat{e}_x$$

$$\nabla^2 = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(\frac{h_2h_3}{h_1}\frac{\partial}{\partial x_1}) + \frac{\partial}{\partial x_2}(\frac{h_1h_3}{h_2}\frac{\partial}{\partial x_2}) + \frac{\partial}{\partial x_3}(\frac{h_1h_2}{h_3}\frac{\partial}{\partial x_3})\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{\partial}{\partial\phi}(\frac{1}{r}\frac{\partial}{\partial\phi}) + \frac{\partial}{\partial x}(r\frac{\partial}{\partial x})\right]$$

$$\nabla^2\vec{V} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r\frac{\partial\vec{V}}{\partial r}) + \frac{\partial}{\partial\phi}(\frac{1}{r}\cancel{\frac{\partial\vec{V}}{\partial\phi}}) + \frac{\partial}{\partial x}(r\cancel{\frac{\partial\vec{V}}{\partial x}})\right] \quad , \quad \vec{V} = u(r)\hat{e}_x$$

$0\ (\vec{V} = u(r)\vec{e}_r)\quad 0$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})\right]\hat{e}_x$$

∴ Momentum equation:

x-dir: $$0 = -\frac{\partial p}{\partial x} + \frac{\mu}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) \qquad (3.5)$$

r-dir: $\frac{\partial p}{\partial r} = 0 \Rightarrow \boxed{p=p(x)}$

Eq. (3.5) $\Rightarrow$

$$\underbrace{\frac{dp}{dx}}_{f_n(x)} = \underbrace{\frac{\mu}{r}\frac{d}{dr}(r\frac{du}{dr})}_{f_n(r)} = \text{constant}$$

∵ L.H.S $= f_n(x)$
R.H.S $= f_n(r)$
∴the only solution is that it is a constant

integrate twice with the B.C.'s: (i) u(r) = 0 (ii) $\left.\frac{du}{dr}\right|_{r=0} = 0$, we obtain

$$\boxed{u(r) = -\frac{1}{4\mu}\frac{dp}{dx}(R^2 - r^2)} \qquad \text{(parabolic profile)}$$

$$u_{\max} = u|_{r=0} = -\frac{1}{4\mu}\frac{dp}{dx}R^2$$

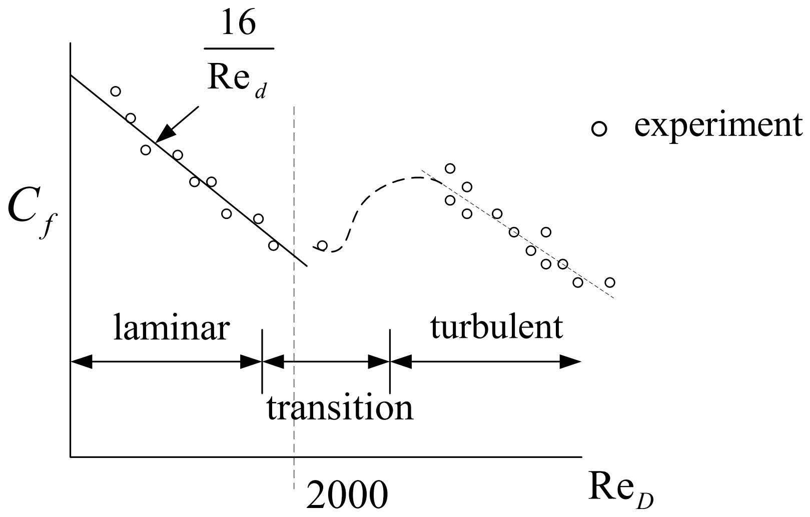
Volume flow rate, $Q = \int_0^R u(r)(2\pi r)dr = -\frac{\pi R^4}{8\mu}\frac{dp}{dx}$

The mean velocity, $\bar{u} = \frac{Q}{\pi R^2} = -\frac{R^2}{8\mu}\frac{dp}{dx} = \frac{u_{\max}}{2}$

Shear stress at wall $\tau_w = \left|\mu\frac{du}{dr}\right| = \frac{1}{2}R(-\frac{dp}{dx}) = \frac{4\mu\bar{u}}{R}$

$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho\bar{u}^2} = \frac{16}{\mathrm{Re}_D}, \text{ where } Re_D = \frac{\rho\bar{u}D}{\mu}$$

which agrees wall with the experiment data for laminar flow



Energy equation:

$$\rho C_v\left[\frac{\partial T}{\partial t} + \vec{V}\cdot\nabla T\right] = -p\nabla\cdot\vec{V} + 2\mu\overleftrightarrow{\varepsilon}:\overleftrightarrow{\varepsilon} + k\nabla^2 T \qquad (2.40)$$

$$\vec{V} = u(r)\hat{e}_x$$

$$\nabla\vec{V} = \frac{du}{dr}\hat{e}_r\hat{e}_x \quad , \quad (\nabla\vec{V})^t = \frac{du}{dr}\hat{e}_x\hat{e}_r$$

$$\overleftrightarrow{\varepsilon} = \frac{1}{2}\left[\nabla\vec{V} + (\nabla\vec{V})^t\right] = \frac{1}{2}\frac{du}{dr}(\hat{e}_r\hat{e}_x + \hat{e}_x\hat{e}_r)$$

$$\overleftrightarrow{\varepsilon}:\overleftrightarrow{\varepsilon} = (\frac{1}{2}\frac{du}{dr})^2\left[\hat{e}_r\hat{e}_x + \hat{e}_x\hat{e}_r\right]:\left[\hat{e}_r\hat{e}_x + \hat{e}_x\hat{e}_r\right]$$

$$= (\frac{1}{2}\frac{du}{dr})^2 \left[\cancel{\hat{e}_r\hat{e}_x : \hat{e}_r\hat{e}_x} + 2\underbrace{\hat{e}_x\hat{e}_r : \hat{e}_r\hat{e}_x}_{\substack{=(\hat{e}_x\cdot\hat{e}_x)(\hat{e}_r\cdot\hat{e}_r)\\ =1}} + \cancel{\underbrace{\hat{e}_x\hat{e}_r : \hat{e}_x\hat{e}_r}_{\substack{=(\hat{e}_r\cdot\hat{e}_x)(\hat{e}_r\cdot\hat{e}_x)\\ =0}}} \right]$$

$$= \frac{1}{2}(\frac{du}{dr})^2$$

$\nabla\cdot\vec{V} = 0$ (Incompressible flow)

$$\vec{V}\cdot\nabla T = (u(r)\hat{e}_x)\cdot(\frac{\partial T}{\partial x}\hat{e}_x + ..) = u\frac{\partial T}{\partial x}$$

$$\nabla^2 T = \frac{1}{r}\left\{\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) + \frac{\partial}{\partial \phi}(\frac{1}{r}\frac{\partial T}{\partial \phi}) + \frac{\partial}{\partial x}(r\frac{\partial T}{\partial x})\right\}$$

Assume $T = T(r)$ only then Eq. (2.40) becomes

$$0 = \mu(\frac{du}{dr})^2 + \frac{k}{r}\frac{\partial}{\partial r}(r\frac{dT}{dr}) \qquad \text{(fully-developed in temperature)}$$

Sub. $(\frac{du}{dr})$ into the above equation, and integrate twice with the B.C.'s:

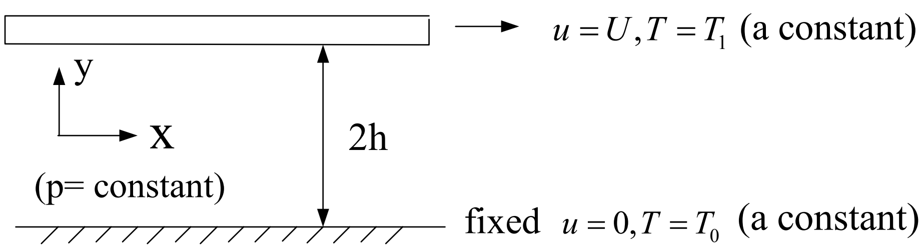
① $T(r) = T_w$ ② $\left.\frac{dT}{dr}\right|_{r=0} = 0$, we have

$$\boxed{T(r) = T_w + \frac{1}{64k\mu}(\frac{dp}{dx})^2(R^4 - r^4)}$$

Remark:

① Can discuss $q_{wall} \sim (\frac{dp}{dx})^2$, while $\tau_w \sim (\frac{dp}{dx})$ and $Q \sim (\frac{dp}{dx})$, $Q \sim R^4$

c) Couette (Wall Driven) Duct Flow:



continuity: $\frac{\partial u}{\partial x} = 0$

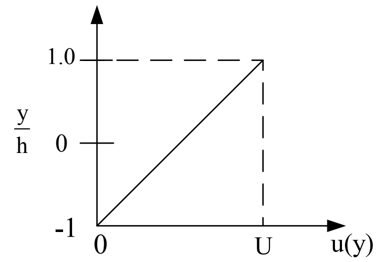
momentum: $0 = \mu\frac{d^2u}{dy^2}$

Since the plate is infinite long with constant wall temperature, the temperature can be assumed fully developed. Thus T=T(y) only. The energy equation reduces to

$$0=\mu\left(\frac{du}{dy}\right)^2+k\frac{d^2T}{dy^2}$$

From momentum equation & B.C.'s, we have velocity distribution

$$u(y)=\frac{U}{2}\left(1+\frac{y}{h}\right)$$



shear stress at any point.

$$\tau=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=\frac{\mu U}{2h}=const$$

$C_f \equiv$ function coefficient

$$\equiv\frac{\tau}{\frac{1}{2}\rho U^2}=\frac{\mu}{\rho Uh}=\frac{1}{\mathrm{Re}_h}\quad\text{, where}\quad \mathrm{Re}_h=\frac{\rho Uh}{\mu}$$

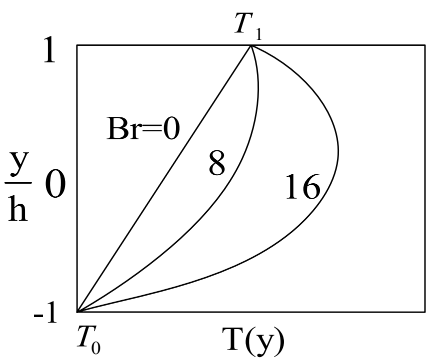
Knowing $\frac{\partial u}{\partial y}$, we can get T(y) from energy equation & B.C.'s:

$$T(y)=\underbrace{\left[\frac{T_1+T_0}{2}+\frac{T_1+T_0}{2}\frac{y}{h}\right]}_{\text{(Due to conduction of fluid)}}+\underbrace{\frac{\mu U^2}{8k}\left(1-\frac{y^2}{k^2}\right)}_{\text{(Due to viscous dissipation)}}$$

Define: Brinkman Number, Br

$$Br\equiv\frac{\mu U^2}{k(T_1-T_0)}=\frac{dissipation\ effect}{conduction\ effect}$$

$$=\frac{\mu C_p}{k}\frac{U^2}{C_p(T_1-T_0)}=\Pr Ec$$



(Br=0 means the flow is pure conduction since dissipation effect is zero)

Question:

From velocity profile, the $u_{\max}$ occurs at $y = h$ (upper plate) and τ is constant at any point. It looks that the viscous dissipation should have equal magnitude every where or at least near the upper plate. But as we can see from temperature profile, the $T_{\max}$ does not occur at hot upper plate, why? Explain this from physical phenomena?

Answer:

Energy dissipation is independent of *y*, as well as τ. But since the wall temperature is different, therefore, q_{upper} is lower while q_{lower} is higher as can be seen from q_{wall} on next page. Thus, $T_{\max}$ occurs in upper half region.

As the given example in p.108 of white, except for giving oils, we commonly neglect dissipation effect in low speed flow temperature analysis. (∵Br is very small)

Heat transfer at the walls:

$$q_w = \left| k \frac{\partial T}{\partial y} \right|_{\pm h} = \frac{k}{2h}(T_1 - T_0) \pm \frac{\mu U^2}{4h} \tag{*}$$

the heat convection coefficient, h_c, is defined as

$$h_c = \frac{q_w}{\triangle T} = \frac{q_w}{T_1 - T_0} \tag{**}$$

Define

$$\text{Nusselt} \quad N_0 \equiv N_u \equiv \frac{h_c L}{k}$$

Take characteristic length L=2h, we have

$$N_u = \frac{h_c(2h)}{k} = 1 \pm \frac{Br}{2}$$

Since *Br* =0 means the dissipation effect is zero, the flow is pure conduction heat transfer. (*Nu* = 1+0 = 1 here) Thus, the numerical value of *Nu* represents the ratio of convection heat transfer to conduction for the same value of $\triangle T$.

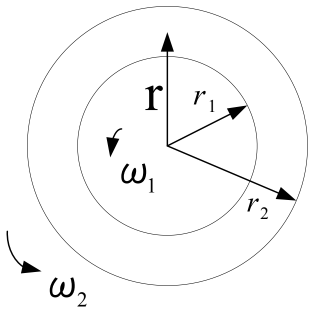
d) Couette (Wall Driven) Pipe Flow

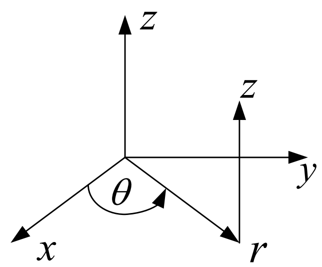
For the flow between two concentric cylinders rotating at angular velocity w_1 and w_2, the fluid has velocity of

$$\vec{V} = u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_z \hat{e}_z$$

Assume:

$$\begin{cases} \left.\begin{array}{l} u_r = u_z = 0 \\ u_\theta = u(r) \end{array}\right\} & \text{paralle} \\ \left.\begin{array}{l} p = p(r) \\ T = T(r) \end{array}\right\} & \text{2-D + fully-developed} \\ \rho = \text{constant} & \end{cases}$$





The continuity equation is identically satisfied. The momentum equation can be reduced as

$$\frac{\rho u^2}{r} = \frac{dp}{dr} \qquad \text{(in r-dir)} \qquad (3.6a)$$

and

$$\frac{d^2u}{dr^2} + \frac{d}{dr}\left(\frac{u}{r}\right) = 0 \qquad (\text{in } \theta\text{-dir}) \qquad (3.6b)$$

With the B.C.'s: (i) $u(r_1) = w_1 r_1$

(ii) $u(r_2) = w_2 r_2$

Eq. (3.6b) becomes

$$u(r) = \frac{1}{r_2^2 - r_1^2}\left\{ r(w_2 r_2^2 - w_1 r_1^2) - \frac{r_1^2 r_2^2}{r}(w_2 - w_1) \right\} \qquad (3.7)$$

<u>Remarks</u>:

(1) If $r_2 \to \infty$, $w_2 \to 0$

(i) $r_2^2 - r_1^2 \approx r_2^2$

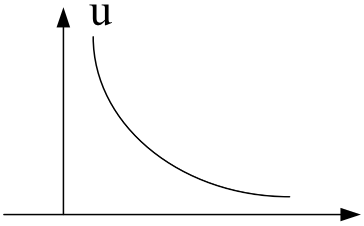
(ii) $w_2 \to 0$, $r_2 \to \infty$ $\quad \therefore w_2 r_2 \to$ uncertain No: however

$w_2 r_2^2 \to \infty$, $\therefore w_2 r_2^2 - w_1 r_1^2 \approx w_2 r_2^2$

$$u(r) \approx \frac{1}{r_2^2}\left\{ r w_2 r_2^2 + \frac{r_1^2 r_2^2}{r} w_1 \right\} = r w_2 + \frac{r_1 w_1}{r} = \frac{r_1^2 w_1^2}{r}$$

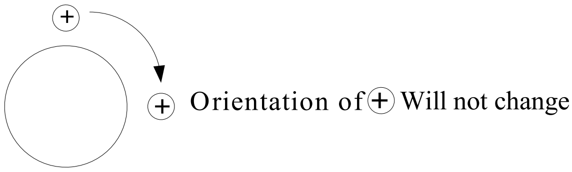
$$\Gamma \equiv circulation \equiv \oint u(r)\, d\ell = (\frac{r_1^2 w_1}{k})(2\pi r) = 2\pi w_1 r_1^2 = const$$

$$\therefore u(r) = \frac{\Gamma_0}{2\pi r}$$



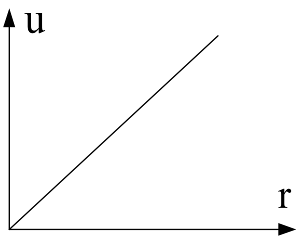
potential vortex (free vortex)

$\because \nabla \times \vec{V} = 0$,No vorticity



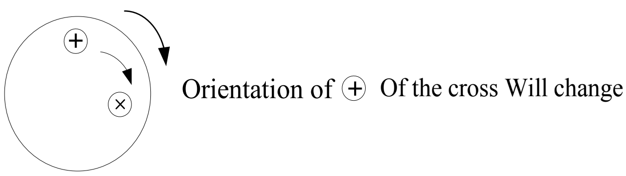
(2) If $r_1 = w_1 = 0$ (No inner cylinder)

$$u(r) = \frac{1}{r_2^2}\left\{ r w_2 r_2^2 \right\} = r w_2$$



Rigid body rotation (froce vortex)

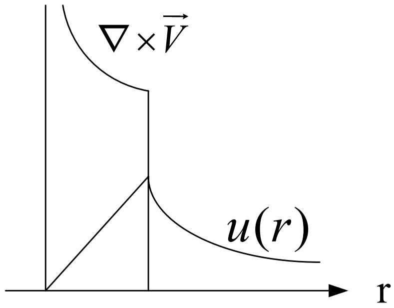
$\because \nabla \times \vec{V} \neq 0$



$$\nabla \times \vec{V} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\phi & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & rw_2 & 0 \end{vmatrix} = \frac{1}{r}\left[\frac{\partial (rw_2)}{\partial r}\right] = \frac{w_2}{r} = \Omega$$

$$\therefore \Omega\big|_{r=0} \to \infty$$

(3) A 〝Tornado〞 is a combination of potential vortex & Rigid-body rotation.



The viscous stress of the fluid can be stress as

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[\frac{du}{dr} - \frac{u}{r}\right] \tag{3.8}$$

The moment on the outer cylinder of unit height is

$$\vec{M}_2 = \oint\!\!\!\oint_{S2} \vec{r} \times (\hat{n} \cdot \vec{\tau})\, ds$$

By Eq. (3.8), we can show that

$$\boxed{M_2 = 4\pi\mu \frac{r_1^2 r_2^2 (w_2 - w_1)}{r_2^2 - r_1^2}} \tag{3.9}$$

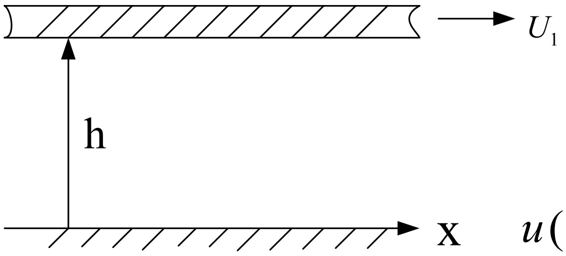
From the energy equation; we can derive the temperature distribution as

$$\boxed{T(r) = \frac{\alpha}{r^2} + \frac{1}{\ell n(r_2/r_1)}\left[(T_1 - \frac{\alpha}{r_1^2})\ell n\frac{r_2}{r_1} + (T_2 - \frac{\partial}{r_2^2})\ell n\frac{r}{r_1}\right]} \tag{3.10}$$

where $\alpha = -\frac{\mu}{k}\left[\frac{r_1^2 r_2^2 (w_1 - w_2)}{r_2^2 - r_1^2}\right]^2$

<u>Remark</u>: The derivation of Eqs. (3.6)~(3.10) can be left as a homework problem for the students.

e) Combined Couette and Poiseuilli Duct Flow



$u(h) = U_1, T(h) = T_1$

$u(0) = 0, T(0) = T_0$

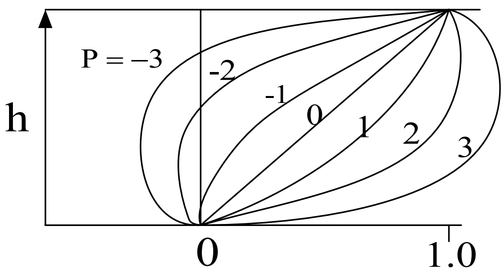
Then the solution of the momentum of (3.4) becomes

$$u(y) = \frac{U_1}{h} y - \underbrace{\frac{h^2}{2\mu}\frac{dp}{dx}}_{\equiv U_1 \mathrm{P} = pressure\ gratient\ \ parameter} \frac{y}{h}(1 - \frac{y}{h})$$

or

$$\frac{u}{U_1} = \frac{y}{h} + \mathrm{P}\frac{y}{h}(1 - \frac{y}{h})$$

The velocity profile is:



As $\mathrm{P} < -1$
;backflow occurs.
This is called the separation of the flow

The function along the upper & lower

$$\tau_w^{\pm} = \mu(\frac{du}{dy})_{y=0,h} \quad , \quad C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho U_1^2}$$

We have

$$\begin{cases} (C_f)_{y=0} = \dfrac{2}{\mathrm{Re}}(1 + \mathrm{P}), \text{ where } \ \mathrm{Re} = \dfrac{\rho U_1 h}{\mu} \\ (C_f)_{y=h} = \dfrac{2}{\mathrm{Re}}(1 - \mathrm{P}) \end{cases}$$

$$\therefore C_f = C_f(\mathrm{Re}, \mathrm{P})$$

In general:

$$C_f = C_f(\mathrm{Re},\ \mathrm{P},\ \underbrace{\frac{T_w^+}{T_w^-}},\ \underbrace{\frac{\mu C_p}{k} = \mathrm{Pr}})$$

(If consider conductivity)

If the flow is compressible, connected with the energy equation

For the energy equation, if we assume also T=T(y) only, then

$$\frac{T-T_0}{T_1-T_0} = \frac{y}{h} + \frac{1}{2}\underbrace{\frac{\mu U_1^2}{h(T_1-T_0)}}\frac{y}{h}(1-\frac{y}{h})$$

$$= \frac{\mu C_p}{k} \times \frac{U_1^2}{C_p(\triangle T)_0} \equiv \mathrm{Pr}\cdot Ec$$

Where ($\triangle T_0 \equiv T_1 - T_0$)

$$\mathrm{Pr} \equiv \frac{\mu C_p}{k} \equiv \text{Prandtl } No. = \frac{\textit{viscous diffusion rate}}{\textit{thermal diffusion rate}}$$

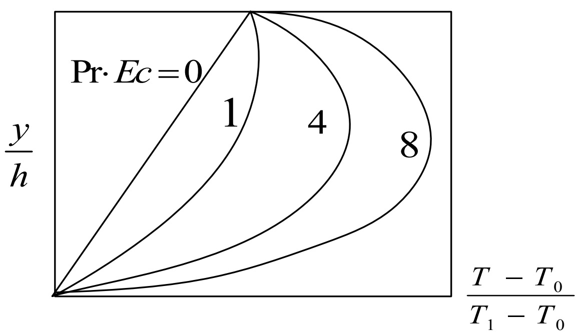
$$Ec \equiv \frac{U_1^2}{C_p(\triangle T)_0} \equiv \text{Eckert No.} = \frac{2(\triangle \mathrm{T})_{\mathrm{ad}}}{(\triangle T)_0} = (r-1)M^2\frac{T_\infty}{(\triangle T)_0},\ \mathrm{M} \rightarrow \text{Mach No.}$$

$\approx$ work of compression (or the absolution temperature of the free stream)/(temperature difference)

(Ec is important when the velocity is comparable with sound speed.)

also $\eta = \dfrac{y}{h}$

then $\boxed{\dfrac{T-T_0}{T_1-T_0} = \eta + \dfrac{1}{2}\mathrm{Pr}\cdot Ec\ \eta(1-\eta)}$



3.2 Simple Unsteady Flow

Assumptions: (1) Constant density, $\rho = \text{constant}$

(2) μ , λ , k constants

(3) planar parallel flow $\frac{\partial}{\partial z} = 0$, $v = w = 0$

With the assumption above, we can only have

$$u = u(t,x,y)\,,\quad p = p(t,x,y)\,,\quad T = T(t,x,y)$$

$$\left\{\begin{array}{l} \text{By continuity equation: } \frac{\partial u}{\partial x} = 0 \;\Rightarrow\; u = u(t,y) \\ \text{By y-momentum equation: } \frac{\partial p}{\partial y} = 0 \;\Rightarrow\; p = p(t,x) \\ \text{By x-momentum \& continuity: } \frac{\partial^2 p}{\partial x^2} = 0 \;\Rightarrow\; \frac{\partial p}{\partial x} = fn(t) \text{ only} \end{array}\right\}$$

$$\therefore \boxed{\frac{\partial p}{\partial x} = fn(t)} \tag{3.11}$$

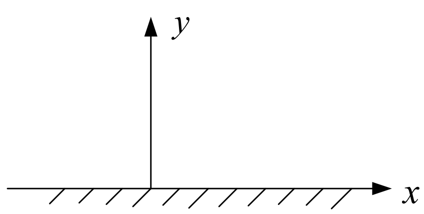
The x-momentum and energy equations become

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{3.12}$$

$$C_v\rho\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x}\right) = \mu\left(\frac{\partial u}{\partial y}\right)^2 + k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{3.13}$$

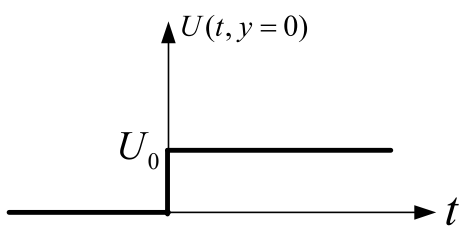
3.2.1 Stokes First problem (Rayleigh's problem)

Consider a semi-infinite space, $y \geq 0$



The air (or any medium) is still for $y \geq 0$, $t < 0$.

At t = 0, become wall impulsively moves to speed U_0



The question is: What is the subsequent motion for y>0, and t>0 ?

First of all, we notice that there is 2 equations ((3.12) & (3.13)) but 3 unknown (p, u, T). One unknown should be removed first. From Eq. (3.11), we have get $\frac{dp}{dx} = fn(t)$ only, i.e. at a certain time (a fixed time) the value of $\frac{dp}{dx}$ is not a function of position, or the value of $\frac{dp}{dx}$ is the same at any point of the flow domain. For our problem, as $y \to \infty$, u should approach to zero velocity for t >0, therefore,

$$\left.\frac{\partial u}{\partial t}\right|_{y\to\infty} = \left.\frac{\partial u}{\partial y}\right|_{y\to\infty} = \left.\frac{\partial^2 u}{\partial y^2}\right|_{y\to\infty} = 0$$

Eqn (3.12) $\Rightarrow$

$$\left.\frac{\partial u}{\partial t}\right|_{y\to\infty} = -\frac{1}{\rho}\left.\frac{dp}{dx}\right|_{y\to\infty} + \nu\left.\frac{\partial^2 u}{\partial y^2}\right|_{y\to\infty}$$

$$\Rightarrow \left.\frac{dp}{dx}\right|_{y\to\infty} = 0$$

From our arguments that $\frac{dp}{dx}$ is not a function of position, hence

$$\boxed{\frac{dp}{dx} = 0} \quad \text{everywhere} \tag{3.14}$$

And the momentum equation becomes

$$\boxed{\frac{du}{dt} = \nu\frac{\partial^2 u}{\partial y^2}} \tag{3.15}$$

with B.C.'S $u(t, y=0) = U_0 \qquad u(t, y\to\infty) = 0$

Eg. (3.15) is a P.D.E, we try to change it into a O.D.E, which will be easier to be solved. Since ν is a constant, Eq.(3.15) become

$$\frac{\partial u}{\partial(\nu\, t)} = \frac{\partial^2 u}{\partial y^2}$$

the independent variables are νt and y, therefore, a new variable should contain this two parameters. Furthermore, we want the new variable to be dimensionless. In this way, if we also set a new dependent variable to replace the old dependent variable u, the equation will be dimensionless. We therefore can solve the O.D.E easier and once for ever. Thus, define

$$\eta = y^{\alpha}(\nu t)^{\beta} \quad , \quad \frac{u}{u_0} = f(\eta)$$

try the dimensional analysis

$$[\nu] = \left[\frac{\mu}{\rho}\right] = \frac{\left[\tau \big/ \frac{\partial u}{\partial y}\right]}{[\rho]} = \frac{\frac{F}{L^2}\cdot\frac{L}{L/T}}{M/L^3} = \frac{FLT}{M} = \frac{\frac{ML}{T^2}\cdot LT}{M}$$

$$= L^2/T$$

$$[\nu \mathrm{t}] = L^2 \quad , \quad [\mathrm{y}] = L$$

So if want to dimensionlize η, we should Take

$$\partial = 1 \qquad \& \qquad \beta = -\frac{1}{2}$$

Thus $\boxed{\eta = \frac{y}{2\sqrt{\nu \mathrm{t}}}}$ (The coefficient 2 in the denominator is taken to make the final O.D.E. more easier to be integrated) (3.16)

Recall

$$\boxed{\frac{u}{u_0} = f(\eta)} \tag{3.17}$$

$$u = U_0 f(\eta)$$

$$\begin{cases} \dfrac{\partial \eta}{\partial t} = -\dfrac{1}{2t}\dfrac{y}{2\sqrt{\nu \mathrm{t}}} = \dfrac{-\eta}{2t} \quad , \quad \dfrac{\partial \eta}{\partial y} = \dfrac{1}{2\sqrt{\nu \mathrm{t}}} \\[2ex] \dfrac{\partial u}{\partial t} = U_0 \dfrac{df}{d\eta}\dfrac{\partial \eta}{\partial t} = -U_0 \dfrac{df}{d\eta}\dfrac{\eta}{2t} \\[2ex] \dfrac{\partial u}{\partial y} = U_0 \dfrac{df}{d\eta}\dfrac{\partial \eta}{\partial y} = U_0 \dfrac{df}{d\eta}\dfrac{1}{2\sqrt{\nu \mathrm{t}}} \\[2ex] \dfrac{\partial^2 u}{\partial y^2} = U_0 \dfrac{d^2 f}{d\eta^2}\dfrac{1}{4\nu \mathrm{t}} \end{cases}$$

Eq. (3.15) $\Rightarrow$

$$-U_0 \frac{df}{d\eta}\frac{\eta}{2t} = \nu\, U_0 \frac{d^2 f}{d\eta^2}\frac{1}{4\nu\, t}$$

$$\Rightarrow \qquad \boxed{\frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} = 0} \tag{3.18}$$

with B.C'S: ① $\eta = 0$, $f(0) = 1$

② $\eta \to \infty$, $f(\infty) = 0$

Eq. (3.18) $\Rightarrow$

$$\frac{df'}{f'} + 2\eta f' = 0$$

$$\Rightarrow \frac{df'}{f'} = -2\eta d\eta$$

$$\Rightarrow \ell n\, f' = -\eta^2 + C_1$$

or $$f' = Ae^{-\eta^2}$$

integrate again

$$f = A\int e^{-\eta^2} d\eta + B$$

① $\eta = 0$, $f(0) = 1$

$$1 = \mathrm{A}\int_0^0 \mathrm{e}^{-\eta^2} d\eta + B \qquad \therefore\ \mathrm{B}=1$$

② $\eta \to \infty$, $f(\infty) = 0$

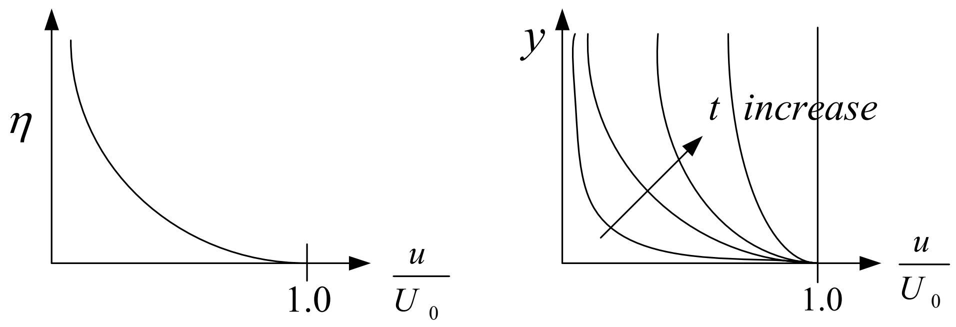
$$0 = \mathrm{A}\int_0^\infty \mathrm{e}^{-\eta^2} d\eta + 1 \qquad \therefore A = \frac{-1}{\int_0^\infty \mathrm{e}^{-z^2} dz} = \frac{-1}{\sqrt{\pi}/2}$$

$$\therefore \qquad f(\eta) = 1 - \frac{2}{\sqrt{\pi}}\int e^{-\eta^2} d\eta$$

or $$\boxed{f(\eta) = 1 - \underbrace{\frac{2}{\sqrt{\pi}}\int_0^\eta e^{-z^2} dz}_{erf(\eta)}} \tag{3.19a}$$

$$\therefore \qquad \boxed{f(\eta)} = 1 - erf(\eta) \equiv \boxed{erfc(\eta)} \tag{3.19b}$$

Complementary error function

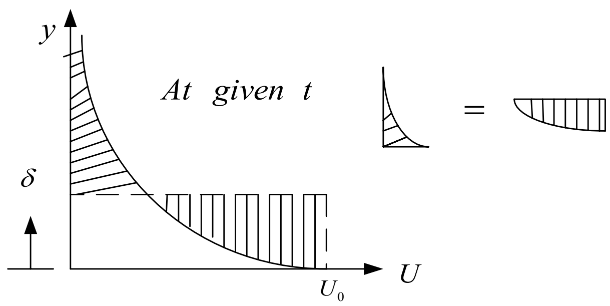


$$\frac{du}{dy} = U_0 \frac{df(\eta)}{d\eta}\frac{d\eta}{dy} = U_0(-\frac{2}{\sqrt{\pi}}e^{-\eta^2})(\frac{1}{2\sqrt{\nu t}}) = \frac{-U_0}{\sqrt{\pi\nu t}}e^{-\eta^2}$$

$$\tau_w = \mu \left.\frac{du}{dy}\right|_{y=0} = -\frac{\mu U_0}{\sqrt{\pi \nu t}} \tag{3.20}$$

The displacement thickness $\delta(t)$ is defined as

$$U_0\delta(t) = \int_0^\infty u(y,t)dy$$

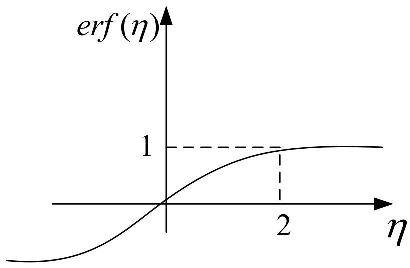


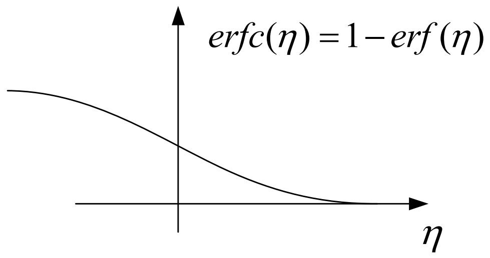
or $$\delta(t) = \int_0^\infty \frac{u}{U_0}(y,t)dy = \int_0^\infty erfc(\eta)\frac{dy}{d\eta}d\eta$$

$$= 2\sqrt{\nu t}\underbrace{\int_0^\infty erfc(\eta)d\eta}$$

integrate by parts

$$\Big(\underbrace{\eta\, erfc\, \eta\Big|_0^\infty}_{0} - \int_0^\infty \eta \frac{d}{d\eta}(erfc\,\eta)d\eta\Big)$$





$$\frac{d(erfc(\eta))}{d\eta} = \frac{df(\eta)}{d\eta} = -\frac{2}{\sqrt{\pi}} e^{-\eta^2}$$

$$\therefore \delta(t) = -2\sqrt{\nu\, t}\int_0^\infty \eta\left(-\frac{2}{\sqrt{\pi}} e^{-\eta^2}\right) d\eta$$

$$= 2\sqrt{\frac{\nu\, t}{\pi}}\int_0^\infty 2\eta e^{-\eta^2}\, d\eta = 2\sqrt{\frac{\nu\, t}{\pi}}\int_0^\infty e^{-\eta^2}\, d(\eta^2)$$

$$= 2\sqrt{\frac{\nu\, t}{\pi}}\left[-e^{-\eta^2}\right]_0^\infty = 2\sqrt{\frac{\nu\, t}{\pi}} \qquad (3.21)$$

for example:

Kg/m s

at t=10 sec

	ν (m^2/s)	δ (m)	μ (Pa Sec)
Air,40℃	1.71×10^{-5}	0.0147	17.1
Water,℃	6.61×10^{-7}	0.0029	655
Lubricating Oil,℃	1×10^{-4}	0.0357	----------------

Remark: At the first glance, it seems strength that the strength of the momentum transport (or the speed of the propagation of the external disturbance) in three different fluid is:

Oil > Air > water

While the μ of there is in the order of

Oil > water > Air

However, it is reasonable, since $\delta \sim \nu^{1/2} \sim \sqrt{\frac{\mu}{\rho}}$, not only depend on μ.

How about the temperature change if we imposed suddenly a temperature to the boundary? Similarly, we will obtain

$$\delta_T(t) = 2\sqrt{\frac{\alpha t}{\pi}} \tag{3.22}$$

where

$$\alpha = \frac{\kappa}{\rho Cp}$$

$$\therefore \frac{\delta_\mu(t)}{\delta_T(t)} = \sqrt{\frac{\nu}{\alpha}} = \sqrt{\Pr} \qquad (\Pr = \frac{\mu Cp}{k}) \tag{3.23}$$

Remarks:

(1) as Pr>1 , the δ_μ is larger than δ_T

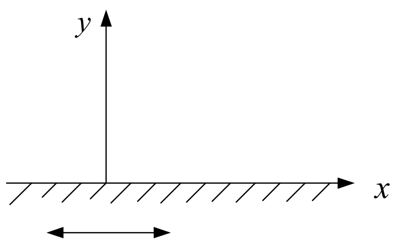
(2) Typical values of Pr for different fluid are

Fluid	Mercury	He	Air	F-12	Methyl alcohol(甲醇)	Water	Ethyl alcohol(乙醇)
Pr	0.025	0.7	0.72	3.7	6.8	7.0	16

Fluid	SAE 30 oil
Pr	3500

(The $\frac{\delta_\mu}{\delta_T}$ are in the order of Air < Water < Oil, now!)

3.2.2 Stokes Second Problem---Oscillating plate



Governing equation:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{3.24}$$

B.C.'s : $u\ (y = 0,\ t) = U_0 \cos \omega t$

$u\ (y \to \infty,\ t) = 0$

It is convenient (and make the procedure easier) to use a complex variable to solve the problem. Furthermore, if we are doing the problem of $u(0,t) = U_0 \sin wt$, we can take the imaginary part of the solution and it is no need to do the problem twice.

$$\because\ e^{i\omega t} = cos\omega t + i \sin t$$

we take the B.C. as

$$u(0,t) = U_0 e^{i\omega t} \tag{3.25}$$

Use separation of variables, we assume

$$u(y,t) = U_0 e^{i\omega t} f(y) \tag{3.26}$$

$$\left(\begin{array}{l}\text{Note : The solution fo this problem will be the real part of Eg.(3.26).}\\ \qquad\text{Which is the solution under the B.C. of Eg.(3.25)}\end{array}\right)$$

$$\frac{\partial u}{\partial t} = i\omega U_0 e^{i\omega t} f$$

$$\frac{\partial u}{\partial y} = U_0 e^{i\omega t} f', \frac{\partial^2 u}{\partial y^2} = U_0 e^{i\omega t} f''$$

sub into egn.(3.24) yields:

$$i\omega U_0 e^{i\omega} f = \nu\ U_0 e^{i\omega t} f''$$

$$f'' - \frac{i\omega}{\nu} f = 0 \tag{3.27}$$

Use characteristic equation to solve, i.e. we assume

$$f = e^{\lambda y} \Rightarrow f' = \lambda e^{\lambda y}, f'' = \lambda^2 e^{\lambda y}$$

sub into Eg (3.27)

$$\lambda^2 - \frac{i\omega}{\nu} = 0 \Rightarrow \lambda = \pm\sqrt{\frac{\omega}{\nu}}\sqrt{i}$$

$$\because \sqrt{i} = \left[e^{i\frac{\pi}{2}}\right]^{1/2} = e^{i\frac{\pi}{4}} = cos\frac{\pi}{4} + i\ sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}(1+i)$$

$$\therefore \lambda = \pm\sqrt{\frac{\omega}{2\nu}}(1+i)$$

(3.26)$\Rightarrow$

$$u(y,t) = U_0 e^{i\omega t} e^{\lambda y} = U_0\left[Ae^{i\omega t + \sqrt{\frac{\omega}{2\nu}}(1+i)y} + Be^{i\omega t} e^{-\sqrt{\frac{\omega}{2\nu}}(1+i)y} \right]$$

$$= U_0\left[\underbrace{Ae^{\sqrt{\frac{\omega}{2\nu}}y} e^{i(\omega t + \sqrt{\frac{\omega}{2\nu}}y)}}_{\left(\begin{array}{l}\text{as } y\to\infty, e^{\sqrt{\frac{\omega}{2\nu}}y}\to\infty \\ \text{but } u(\infty,t)=0, \therefore A=0\end{array}\right)} + Be^{-\sqrt{\frac{\omega}{2\nu}}y} e^{i(\omega t - \sqrt{\frac{\omega}{2\nu}}y)} \right]$$

Also $u(0,t) = U_0 e^{i\omega t} = BU_0 e^{i\omega t} \Rightarrow B = 1$

Thus

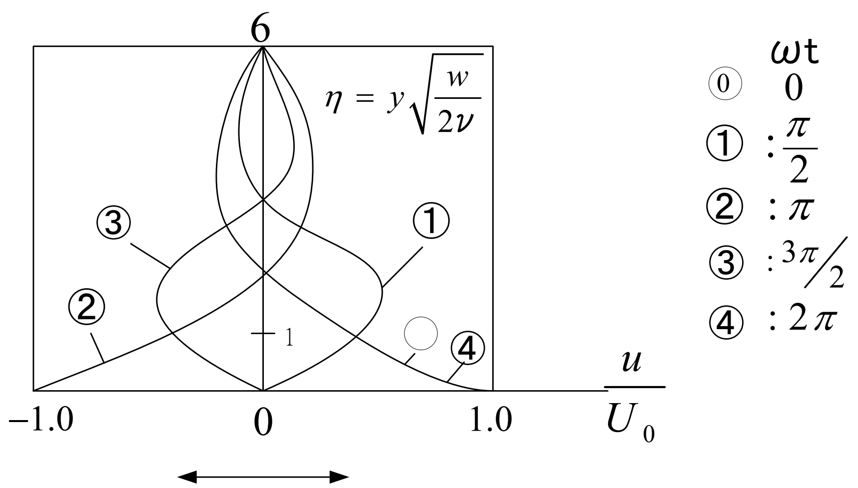
$$u(y,t) = U_0 e^{-\sqrt{\frac{\omega}{2\nu}}y} e^{i(\omega t - \sqrt{\frac{w}{2\nu}}y)}$$

$$= U_0 e^{-\sqrt{\frac{\omega}{2\nu}}y}\left[cos(\omega t - \sqrt{\frac{\omega}{2\nu}}y) + i\ sin(\omega t - \sqrt{\frac{\omega}{2\nu}}y) \right]$$

Since we have only the real part, $\therefore$

$$\boxed{u(y,t) = \underbrace{U_0 e^{-\sqrt{\frac{\omega}{2\nu}}y}}_{\text{Decaying Amplitude}} cos(\omega t - \sqrt{\frac{\omega}{2\nu}}y)} \tag{3.28}$$

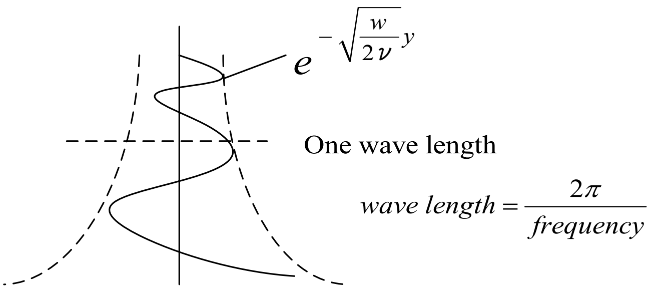
The velocity distribution is



<u>Remarks:</u>

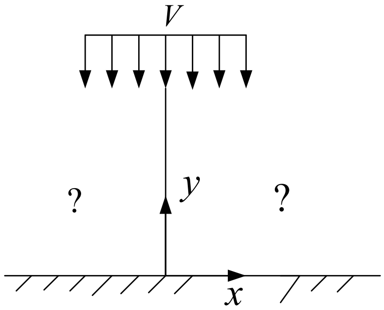
(1) This is similar to the temp. varies on the earth <u>every day</u> due to the sunrise and sunset..
(or, if we take u as the average temp. of a day, the distribution will similar to the temp. on the earth <u>every year</u> due to the seasons.)

(2)



$$\lambda = \frac{2\pi}{\sqrt{\frac{\omega}{2\nu}}} = 2\pi\left(\frac{2\nu}{\omega}\right)^{1/2} \equiv \text{depth of penetration}$$

3.3 steady, 2-D stagnathion flow (Hiemenz Flow)



Z-direction is infinite , but the distributin of $\vec{V}$ in x-span is finite, therefore it will have a stagnation pt on the plate, where we take as the origin of the coord. system. Our objective is to understand the flowfield near the stagnation pt!

For 2-D, steady, incompressible. Flow with constant μ , the G..E'S are:

$$\begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}=0 \\ \rho\left[u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}\right]=-\dfrac{\partial p}{\partial x}+\mu(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}) \\ \rho\left[u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}\right]=-\dfrac{\partial p}{\partial y}+\mu(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}) \end{cases} \qquad (3.29)$$

if we consider a particular solution, say

$$\begin{cases} u=ax \\ v=-ay \end{cases} \qquad (3.30)$$

Continuity equation: a-a = 0 (√)

y-momentum: $\rho\left[0+(-ay)(-a)\right]=-\dfrac{\partial p}{\partial y}\Rightarrow P=\dfrac{-1}{2}a^2y^2+f(\mathrm{x})$

x-momentum: $\rho\left[(ax)(a)+0\right]=-\dfrac{\partial p}{\partial x}\ \Rightarrow P=\dfrac{-1}{2}a^2x^2+g(u)$

$$\therefore P=\frac{-1}{2}\underbrace{a^2y^2}_{v^2}-\frac{1}{2}\underbrace{a^2x^2}_{u^2}+const$$

or $$P+\frac{1}{2}(u^2+v^2)\equiv P=const.$$

This is the Bernoulli equation, that is the given velocity distribution is for a inviscid flow. The streamline is given as:

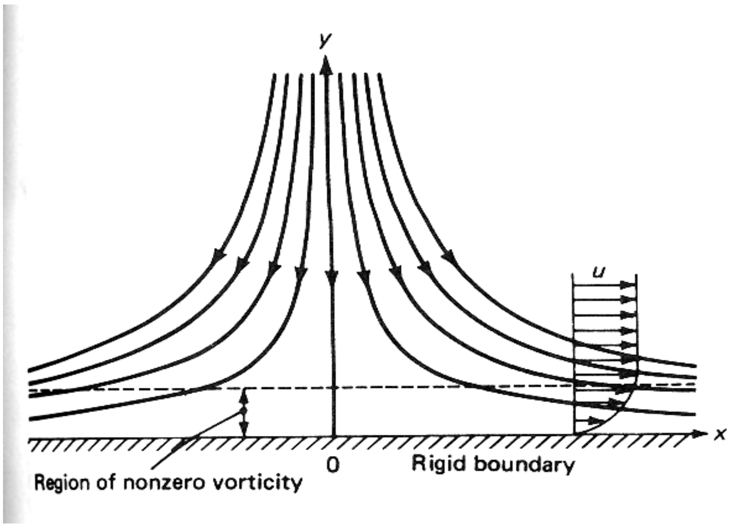
$$\vec{V}\times d\vec{s}=0 \qquad \text{(parallel each other)}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u & v & 0 \\ dx & dy & 0 \end{vmatrix} = (udy - vdx)\vec{k} = 0$$

$$\Rightarrow \quad \frac{dx}{u} = \frac{dy}{v} \quad \Rightarrow \quad \frac{dx}{ax} = \frac{dy}{-ay} \quad \Rightarrow \quad \ell n\ x = -\ell n\ y + C$$

$$\Rightarrow \quad \ell n\ (xy) = C \quad \Rightarrow \quad xy = \text{constant} \qquad \text{family of hyperbalas}$$

Therefore, the streamline looks like:



<u>Remarks</u>:

(1) though the given velocity distribution satisfies the N-S equation, it can't satisfy the no slip B.C'S.
($@\ y = 0,\ v = 0$ but $u = ax \neq 0$, except for $x = 0$

(2) we , therefore , want to modify the u, v, such that it can satisfies the no slips boundary condition

To modify v= -ay, let us assume a similar form of

$$\boxed{v = -f(y)} \tag{3.31a}$$

To satisfy the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial x} - f^{'}(y) \Rightarrow u = xf^{'}(y)$$

or

$$\boxed{u = xf^{'}(y)} \tag{3.31b}$$

In order to satisfy the no-slip B.C'S:

$$u\big|_{y=0} = 0 \Rightarrow f^{'}(0) = 0$$

$$v\big|_{y=0} = 0 \Rightarrow f(0) = 0 \qquad (3.32a, b)$$

As the $y \to \infty$, we want u back to the inviscid case, that is $u = ax$, thus

$$\boxed{f^{'}(\infty) = a} \qquad (3.32c)$$

In the inviscid flow , the pressure is $p = p_0 - \frac{1}{2}\rho\left[a^2x^2 + a^2y^2\right]$

Now, we modify the pressure as

$$P = P_0 - \frac{1}{2}\rho\left[a^2x^2 + a^2F(y)\right] \qquad (3.33)$$

Not that *u, v, p* are replaced by two unknown function $f(y)$ and $F(y)$. However, we still have two momentum equations. the problem is closure.

Sub. *u*, *v*, and *p* into the *x*-momentum equation, we have

$$\boxed{f^{'2} - ff^{''} = a^2 + \nu f^{'''}} \qquad (3.34)$$

Sub *u*, *v* Δp into *y*-momentum equation:

$$\Rightarrow ff^{'} = \frac{1}{2}a^2F^{'} - \nu f^{''}$$

or $$F^{'} = \frac{2}{a^2}\left[\nu f^{''} + ff^{'}\right]$$

or $$\boxed{F = \frac{2}{a^2}\left[\nu f^{'} + \frac{f^2}{2}\right] + const} \qquad (3.35)$$

In summary, we have

$$\nu f^{'''} + ff^{''} - f^{'2} + a^2 = 0 \qquad (3.34)$$

$$F = \frac{2}{a^2}\left[\nu f^{'} + \frac{f^2}{2}\right] + const \qquad (3.35)$$

with B.C'S. ① $f(0) = 0$ ② $f^{'}(0) = 0$ ③ $f^{'}(\infty) = a$

with eq.(3.34) and B.C'S, we can solve the unknown function f. We want to use similarity method, introduce

$$\eta = \alpha y, \qquad f(y) = A\phi(\eta)$$

then

$$\nu A\alpha^3\phi''' + \underbrace{(A\phi)(A\alpha^2\phi''')}_{A^2\alpha^2\phi\phi''} - A^2\alpha^2\phi'^2 + a^2 = 0$$

To let the equation non-dimensionalized, i.e., let the coefficients of the above equation become all identically equal to unity , we put

$$\nu A\alpha^3 = a^2 \quad \text{and} \quad A^2\alpha^2 = a^2$$

$$\therefore \quad A = \sqrt{\nu a} \quad \text{and} \quad \alpha = \sqrt{\frac{a}{\nu}}$$

Thus, the new independent variables are

$$\eta = \sqrt{\frac{a}{\nu}}y \ , \ F(y) = \sqrt{\nu a}\phi(\eta) \tag{3.36}$$

The G.E's become

$$\boxed{\begin{aligned} &\phi''' + \phi\phi'' - \phi'^2 + 1 = 0 \\ &\textit{with}\ \text{B.C}'s: \\ &\phi(0) = 0,\ \phi'(0) = 0,\ \phi'(\infty) = 1 \end{aligned}} \tag{3.37}$$

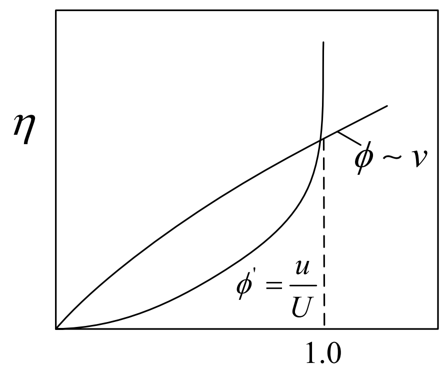
⇐Hiemenz Flow

also get

$$\boxed{F = \frac{\nu}{a}(\phi^2 + 2\phi')} \tag{3.38}$$

Eqn (3.37) is solved by Hiemenz, and tabulate as Table 5.1 in the p.p98 of schlichting.

$\eta = \sqrt{\frac{a}{\nu}}y$	ϕ	$\frac{d\phi}{d\eta} = \frac{u}{U}$
0	0	0
0.2	0.0233	0.2266
:	:	:
2.4	1.7553	0.9905
:	:	:
4.0	3.3521	1.000
:	:	:
4.6	3.9521	1.000



$$\left(\phi' = \frac{d\phi}{d\eta} = \frac{d\left(f/\sqrt{a\nu}\right)}{d\left(\sqrt{\frac{a}{\nu}}y\right)} = \sqrt{\frac{1}{a\nu\cdot\frac{a}{\nu}}}\frac{df}{dy} = \frac{1}{a}f' = \frac{1}{a}\frac{u}{x} = \frac{u}{U}\right)$$

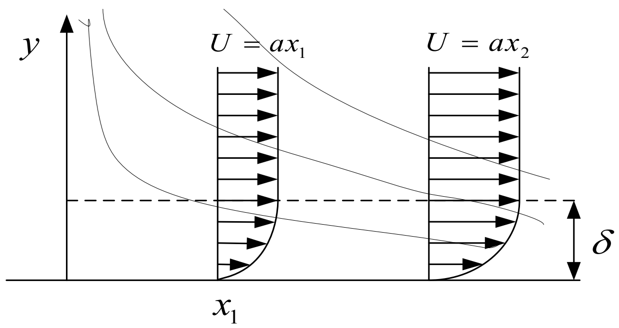
Remarks:

(1) As $\eta = 2.4$, $u/U = 0.9905$. We consider the corresponding distance from the wall as the boundary layer δ, therefore ($\eta = \sqrt{\frac{a}{\nu}}y \quad y = \eta\sqrt{\frac{\nu}{a}}$)

$$\delta = \eta_\delta\sqrt{\frac{\nu}{a}} = 2.4\sqrt{\frac{\nu}{a}} \tag{3.39}$$

Note also that δ is independent of x.

(The boundary-layer thickness is constant because the thinning due to stream acceleration exactly balances the thicknessing due to viscous dissipation)



(2) As $x \to \infty$, $v = -ay$ & $u \to \infty$ for $y \neq 0$

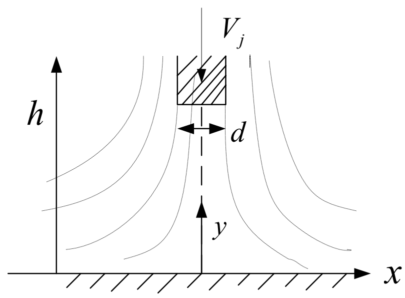
As $y \to \infty$ (or $\eta \to \infty$), $u = ax$ and $v \to \infty$ ($\because \phi \to \infty$)

That is the modified solution, though satisfies the no slip condition, still can't satisfy the condition at infinite. We will see this problem in the "boundary layer theory".

(Localized solution) ⟶ Wrong, the sol satisfies the B.C at infinite. The sol. Match the inviscid flow solution when it is far away from walls.

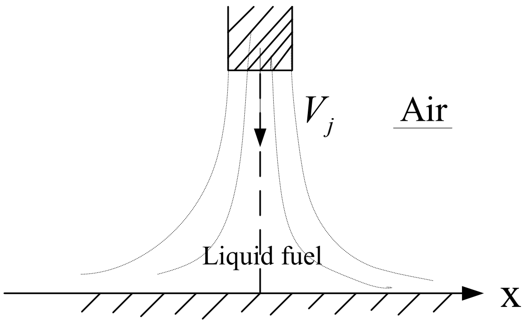
Corresponding problem:

2-D or axis-symmetric stagnation jet:
If jet fluid is the same as the surrounding fluid
How about the flow field? How does it look like?
How we specified the boundary condition?



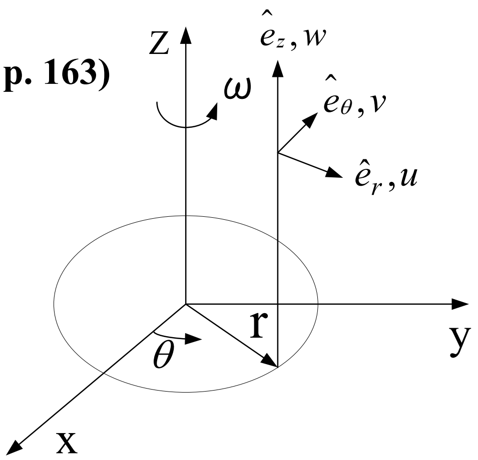
(The potential flow solution $u = ax \quad v = -ay$, which is a ideal, theoretical flow case, will not be the outer solution of the present problem. We need to solve this problem by Numerical method.)

2-D or axis-symmetric Spraying.



How does the spray looked like?

<u>3.4 Flow over a rotating disk</u> (White, 3-8.2, p. 163)



Infinite plane disk rotating with angular velocity ω

Symmetric with respect to $\theta \Rightarrow \dfrac{\partial}{\partial\theta} = 0$

Continuity: $\dfrac{1}{r}\dfrac{\partial}{\partial r}(ru) + \dfrac{\partial}{\partial z}(w) = 0$

r- Momentum: $u\dfrac{\partial u}{\partial r} + w\dfrac{\partial u}{\partial z} - \dfrac{v^2}{r} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r} + \nu\left[\dfrac{\partial^2 u}{\partial r^2} + \dfrac{\partial}{\partial r}\left(\dfrac{u}{r}\right) + \dfrac{\partial^2 u}{\partial z^2}\right]$

θ-momentum: $u\dfrac{\partial v}{\partial r} + w\dfrac{\partial v}{\partial z} + u\dfrac{v}{r} = \nu\left[\dfrac{\partial^2 v}{\partial r^2} + \dfrac{\partial}{\partial r}\left(\dfrac{v}{r}\right) + \dfrac{\partial^2 v}{\partial z^2}\right]$

z- Momentum: $u\dfrac{\partial w}{\partial r} + w\dfrac{\partial w}{\partial z} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} + \nu\left[\dfrac{\partial^2 w}{\partial r^2} + \dfrac{1}{r}\left(\dfrac{\partial w}{\partial r}\right) + \dfrac{\partial^2 w}{\partial z^2}\right]$

4 equations, 4 unknowns (√)

How many B.C's do we need?
--- second order in u, v, w and 1st order in p; thus we need 7 boundary conditions.

B.C.'s:

(1) At $z = 0$, $u = w = 0$, $v = r\omega$ (3)

$p = 0$ (a convenient constant) (1)

(2) At $z = \infty$, $u = v = 0$ (2)

w = ? ($w \neq 0$, because the fluid near the rotating disk will be pumped out, so we expected there are fluid coming from the top of the rotating disk.)

Need one more boundary condition.

(3) $\dfrac{\partial p}{\partial r} = 0$ (so that p is bounded, otherwise $p \rightarrow \pm\infty$ as $r \rightarrow \infty$)

(The flow would move in circular streamlines if the pressure increased radially to balance the inward centripetal acceleration.)

Compare inertial & viscous term in the r-momentum:

$$u\frac{\partial u}{\partial r} \sim \nu \frac{\partial^2 u}{\partial z^2}$$

$$O\left[(\omega r)(\omega)\right] \sim O\left[\nu(\omega r)/\delta^2\right] \Rightarrow \delta \sim (\nu/\omega)^{1/2}$$

Therefore, we may non-dimensional z by the use of δ . Introduce a new variable

$$\zeta = \frac{z}{\delta} = z(\frac{\omega}{\nu})^{1/2} \qquad \textbf{(White: } z^*\textbf{)}$$

Also, try to use separation variables method by assuming

$$\left.\begin{aligned} u &= \omega\, r\, F(\zeta) \\ v &= \omega\, r\, G(\zeta) \end{aligned}\right\} \text{(so that the effect of } r \text{ and } z \text{ are separated; } \boldsymbol{u = v_r;\ v = v_\theta;\ w = v_z})$$

$$w = (\nu\omega)^{1/2} H(\zeta) \leftarrow \text{ function of } z \text{ only since } r \,\&\, z \text{ are assumed separated}$$

$$p = \rho\, \nu\, \omega\, P(\zeta) \qquad \leftarrow \text{ Since } \frac{\partial p}{\partial r} = 0 \quad \therefore p \text{ is function of } z \text{ only})$$

The B.C.'s become:

$$\zeta = 0,\ F(0) = H(0) = P(0) = 0,\ G(0) = 1 \qquad (z = 0)$$

$$\zeta = \infty,\ F(\infty) = G(\infty) = 0 \qquad (z \to \infty) \qquad (3.40)$$

$(\frac{\partial p}{\partial r} = 0$ cancel one term in r-momentum equation!) **$\boxed{3\text{-}185}$**

The G.E.'s becomes

Continuity: $2F + H' = 0$

r: $F^2 - G^2 + HF' = F''$

θ: $2FG + HG' - G'' = 0$

z: $P' + HH' - H'' = 0$ (3.41a~d)

$\boxed{3\text{-}184}$

Equation (3.41a-c) with B.C. (3.40) is sufficient to solve F, H, and G the results can be applied to equation (3.41d) to solve <u>P</u>.

For small value of z, such that ζ is small seek a solution in powers of ζ

$$F = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + \cancel{h.o.T} \qquad \text{neglecting high order terms}$$

$$G = b_0 + b_1\zeta + b_2\zeta^2 + b_3\zeta^3 + \cancel{h.o.T}$$

$$H = c_0 + c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + \cancel{h.o.T} \qquad (3.42)$$

Try to determine $a_0, \ldots\ldots c_3$ (12 unknowns)

From B.C's on $\zeta = 0$, $F = 0 \Rightarrow a_0 = 0$

$G = 1 \Rightarrow b_0 = 1$

$H = 0 \Rightarrow c_0 = 0$

Apply the G.E. at $\zeta = 0$, with $F = H = 0$ and $G = 1$, we have

Continuity: $0 + H'(0) = 0 \Rightarrow H'(0) = 0 \Rightarrow \boldsymbol{c_1 = 0}$

r: $0 - 1 + 0 = F''(0) \Rightarrow F''(0) = -1 \Rightarrow \boldsymbol{a_2 = -\frac{1}{2}}$

θ: $0 + 0 - G''(0) = 0 \Rightarrow G''(0) = 0 \Rightarrow \boldsymbol{b_2 = 0}$

Now differentiate original equations ω.r.t. ζ

$$\begin{aligned} &2F' + H'' = 0 \\ &2FF' - 2GG' + H'F' + HF'' - F''' = 0 \\ &2F'G + 2FG' + H'G' + HG'' - G''' = 0 \end{aligned} \qquad \text{(3.43a-c)}$$

Sub. (3.42) into (3.43) again for, $\zeta = 0$, and use the previous results (i.e. $a_0 = 0$, $b_0 = 0$, $c_0 = 0$, $c_1 = 0$, $a_2 = -1/2$, $b_2 = 0$), we get

$$\begin{cases} 2a_1 + 2c_2 = 0 \\ -2b_1 - 6a_3 = 0 \\ 2a_1 - 6b_3 = 0 \end{cases} \qquad \text{(3.44 a-c)}$$

Differentiate Eq. (3.43a) again and evaluate at $\zeta = 0$:

$$2F'' + H''' = 0$$

(at $\zeta = 0$, $F'' = 2a_2 = -1$, $H''' = 6c_3 + 24c_4\varsigma + \ldots\big|_{\varsigma=0} = 6c_3$)

$$\Rightarrow \quad -2 + 6c_3 = 0 \quad \Rightarrow \quad c_3 = \frac{1}{3}$$

We have get 3+3+1=7 coefficients, therefore 5 unknowns left. However, we have 3 equations (Eq 3.44a-c), thus, we can express 3 unknown (c_2, a_3, b_3) in terms of the other 2 unknowns (a_1, b_1). From (3.44) we have

$$C_2 = -a_1, \quad a_3 = -b_1/3, \quad b_3 = a_1/3$$

The solution thus become

$$\begin{cases} F = a_1\zeta - \dfrac{1}{2}\zeta^2 - \dfrac{b_1}{3}\zeta^3 + \ldots. \\ G = 1 + b_1\zeta + \dfrac{a_1}{3}\zeta^3 + \ldots\ldots \\ H = -a_1\zeta^2 + \dfrac{1}{3}\zeta^3 + \ldots\ldots \end{cases} \tag{3.45}$$

Two unknowns: a_1 & b_1 . Also note that Eq. (3.45) will not suitable for $\zeta \to \infty$, because F, G, H will $\to \infty$

Now, let's look at the equation. At $\zeta \to \infty$, where $F(\zeta) = G(\zeta) = 0$ is the known B.C.'s

Continuity: $2F + H' = 0 \quad \Rightarrow \quad H' = 0 \ \to \ \boxed{H(\zeta) = -C}$ ($\because w < 0$ at $\zeta \to \infty$)

$$\left.\begin{aligned} r: &\quad F^2 - G^2 + HF' = F'' \ \Rightarrow \ HF' = F'' \\ \theta: &\quad 2FG + HG' - G'' = 0 \ \Rightarrow \ HG' = G'' \end{aligned}\right\} \Rightarrow$$

$$\begin{cases} F' \sim e^{H_\infty \zeta} \Rightarrow F'(\zeta) \propto e^{-c\zeta} \Rightarrow F(\zeta) \propto e^{-c\zeta} \\ G' \sim e^{H_\infty \zeta} \Rightarrow G'(\zeta) \propto e^{-c\zeta} \Rightarrow G(\zeta) \propto e^{-c\zeta} \end{cases}$$

Thus, in the far away region, we seek solution of the form of

$$\begin{cases} F = A_1 e^{-c\zeta} + A_2 e^{-2c\zeta} + \ldots\ldots \\ G = B_1 e^{-c\zeta} + B_2 e^{-2c\zeta} + \ldots\ldots \\ H = -C + C_1 e^{-c\zeta} + C_2 e^{-2c\zeta} + \ldots\ldots \end{cases} \tag{3.46}$$

Sub (3.46) into the G.E.s

Continuity:

$$\{2A_1 e^{-C\zeta} + O[e^{-2C\zeta}] + \ldots\} + \{-CC_1 e^{-C\zeta} + O[e^{-2C\zeta}] + \ldots\ldots\} = 0$$

$$\Rightarrow \ e^{-C\zeta}: \quad 2A_1 - CC_1 = 0 \qquad \Rightarrow \ C_1 = 2A_1/C$$

r-momentum:

$$\{A_1 e^{-C\zeta} + A_2 e^{-2C\zeta}\}^2 - \{B_1 e^{-C\zeta} + \ldots\}^2 + [-C + C_1 e^{-C\zeta} + \ldots][-A_1 C e^{-C\zeta}$$

$$-2CA_2 e^{-2C\zeta} + \ldots] = +A_1 C^2 e^{-C\zeta} + 4C^2 A_2 C^2 e^{-2C\zeta} + \ldots$$

$$\Rightarrow \ e^{-2c\zeta}: \quad A_1^2 - B_1^2 - CC_1 A_1 + 2C^2 A_2 - 4C^2 A_2 = 0$$

$$\Rightarrow \ A_2 = -\frac{(A_1^2 + B_1^2)}{2C^2}$$

θ-momentum

$$B_2 = 0 \quad \text{and} \quad C_2 = -\frac{(A_1^2+B_1^2)}{2C^3}$$

The solution near $\zeta \to \infty$ is thus

$$\begin{cases} F = A_1 e^{-C\zeta} + (A_1^2 + B_1^2) / 2C^2 e^{-2C\zeta} + \ldots\ldots \\ G = B_1 e^{-C\zeta} + O[e^{-3C\zeta}] + \ldots\ldots \\ H = -C + \dfrac{2A_1}{C} e^{-C\zeta} - \dfrac{(A_1^2+B_1^2)}{2C^3} e^{-2C\zeta} + \ldots\ldots \end{cases} \tag{3.47}$$

Unknowns: A_1, B_1, C

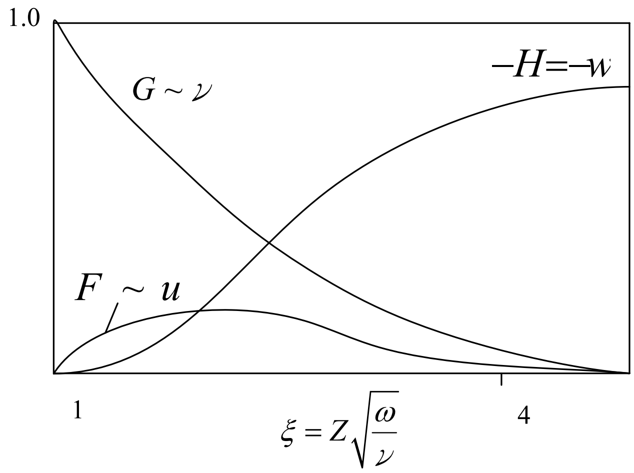
By matching the "inner" solution for small ζ to an "outer" solution for large ζ. That is, take small value of ζ in Eq. (3.47a). Numerically, we finally obtain

$a_1 = 0.51$, $\quad b_1 = -0.616$

$C = 0.886$, $\quad A_1 = 0.934$, $\quad B_1 = 1.208$

These values may not be unique, but they have been verified in laminar flow experiment.

The velocity distribution is

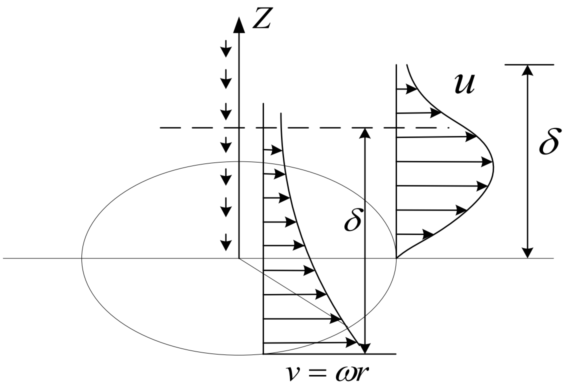


Or , see Table 3.5 on page 166 of the book of White, "Viscous Fluid Flow", 2nd ed.

At $\zeta = 5.4$, $\underline{F \approx G \approx 0.01}$. Therefore the boundary layer δ is

$$5.4 \approx \delta\sqrt{\frac{\omega}{\nu}} \quad \Rightarrow \quad \boxed{\delta = 5.4\sqrt{\frac{\nu}{\omega}}} \tag{3.48} \boxed{(3\text{-}187)}$$

or



see also Fig. 3-28 (White)

- pumping outward near the disk by centrifugal action, replenished from above at <u>constant</u> (at $Z\to\infty$) downward velocity.

<u>Supplementary data for rotating disk:</u>

$\because$ $H(\infty) = -0.8838$, thus

$\boxed{v_z(\infty) = -0.8838\sqrt{\omega\nu}}$ (disk draw fluid toward it)

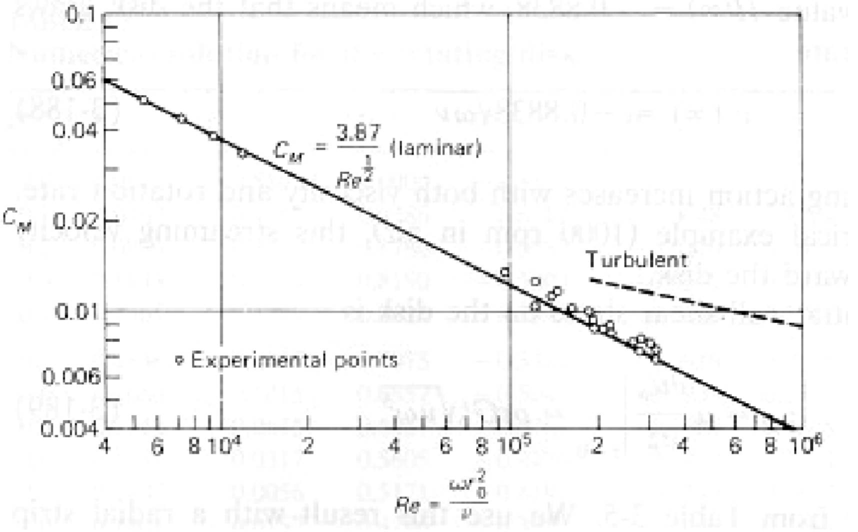
The circumferential wall shear stress on the disk is

$$\tau_{z\theta} = \mu \left.\frac{\partial u_\theta}{\partial z}\right|_{z=0} = \rho\, rG'(0)\sqrt{\nu\omega^3} = -0.6159\rho\, r\sqrt{\nu\omega^3}$$

<u>Remarks</u>:

(1) If we apply the above result, to find the torque required to turn a disk of radius r_0. them

$$M = \int_0^{r_0} \tau_{z\theta} r(2\pi r)dr = -0.967\rho r_0^4\sqrt{\nu\omega^3}\ ;\ C_M \equiv \frac{-2M}{\frac{1}{2}\rho\omega^2 r_0^5} \cong \frac{3.87}{\sqrt{Re}},\ Re = \frac{\omega\, r_0^2}{\nu}$$



The equation agrees well with experimental data for $Re < 3\times10^5$

For $Re < 3\times10^5$, the flow becomes turbulent.

(2) If we stir tea in a cup; the flow pattern will be reversed. Thus these exists an inversed radial flow.

(3) Rogers & Lance (1960) used a Runge-Kutta method to solve eqns (3.41), by defining

$$\begin{cases} Y_1 = H,\ Y_3 = F,\ Y_5 = G \\ Y_2 = F', \qquad Y_4 = G',\ Y_6 = P \end{cases}$$

with I.C's: $Y_1(0) = Y_3(0) = Y_6(0) = 0$, $Y_5(0) = 1$ (3.40a)

The two unknown conditions of $Y_2(0)$ & $Y_4(0)$ must be chosen to satisfy the end B.C.'s (3.40b). Namely

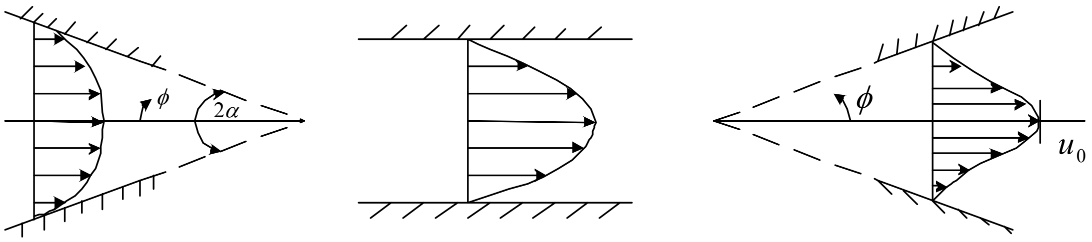
$$Y_3 \to 0,\ Y_5 \to 0, \quad \text{at } \zeta \to \infty.$$

The fortrain statements for (3.41) are simply six statements, as described in p.165 of White's book. By numerical iteration, we can find the I.C.'s to be

$$\begin{cases} Y_2(0) = F'(0) = 0.5102 \\ Y_4(0) = G'(0) = -0.6159 \end{cases}$$

The numerical results agree well with those obtained by asymptotic expansion.

3.5 Flow in a channel (3-8.3 Jeffery-Hamel Flow in a Wedge-Shaped Region)



convergent (sink) *flow*   divergent (source) flow

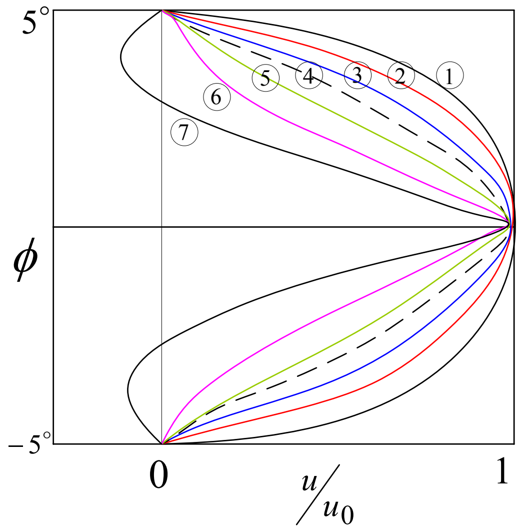
①②③   ④   ⑤⑥⑦

The velocity distributions are

$Re = u_0 r / \nu$

①: $Re = 5000$
②: $Re = 1342$ } convergent
③: $Re = 684$

⑤: $Re = 684$
⑥: $Re = 1342$ } divergent
⑦: $Re = 5000$



3.6 Stream Function

For a 2-D, constant density flow (incompressible flow)

$$\nabla\cdot\vec{V}=0 \;\Rightarrow\; \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \;\Rightarrow\; u=\frac{\partial\psi}{\partial y};\; v=-\frac{\partial\psi}{\partial x} \tag{3.49}$$

$$\Rightarrow\; \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial\psi}{\partial x}\right)=\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial^2\psi}{\partial x\partial y}=0$$

By introducing the stream function "ψ", the continuity equation is automatically satisfied. (By introducing the ψ, the independent variable & the governing equation are reduced by one, however, the order of the P.D.E. increases by one.)

In 3-D flow, the eqn of streamline is

$$\vec{V}\times d\vec{s}=0$$

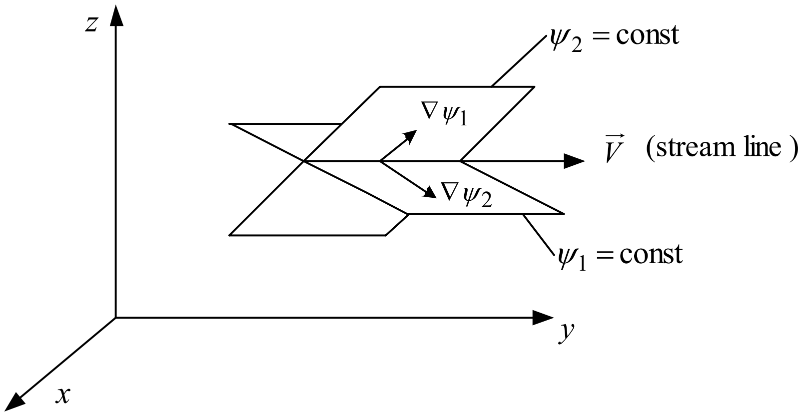
Where $\vec{V}=u\hat{i}+v\hat{j}+w\hat{k}$, $\quad d\vec{s}=dx\hat{i}+dyj+dz\hat{k}$

Thus $\dfrac{dx}{u}=\dfrac{dy}{v}=\dfrac{dz}{w}$

or $\dfrac{dy}{dx}=\dfrac{v(x,y,z)}{u(x,y,z)}$, $\dfrac{dz}{dx}=\dfrac{w(x,y,z)}{u(x,y,z)}$

The stream functions will be

$\psi_1(x,y,z)=C_1=$ constant, $\quad \psi_2(x,y,z)=C_2=$constant



Since $\nabla\psi_1\times\nabla\psi_2$ has the same direction as $\vec{V}$, so we can say

$$\vec{V}=\underbrace{k}_{\text{proportional constant}}(\nabla\psi_1\times\nabla\psi_2)$$

Or

$$\mathbb{K}\vec{V} = \nabla\psi_1 \times \nabla\psi_2$$

Since

$$\nabla\cdot(\nabla\psi_1 \times \nabla\psi_2) = 0 \qquad \text{(Mathematically)}$$

$$\therefore\ \nabla\cdot(\mathbb{K}\vec{V}) = 0$$

But for steady flow, we know $\text{div}(\rho\vec{V}) = 0$, so we can pick up $\mathbb{K} = \rho$, then

$$\boxed{\rho\vec{V} = \nabla\psi_1 \times \nabla\psi_2} \tag{3.50}$$

If the flow is constant density, we know $\text{div}(\rho\vec{V}) = 0$, so we can pick up $\mathbb{K} = 1$, then

$$\boxed{\vec{V} = \nabla\psi_1 \times \nabla\psi_2} \tag{3.51}$$

Remarks:

There are only a few exact solutions for N-S equation unless the physical problem and geometry is easy. The N-S equation may be simplified as $Re \gg 1$ or $Re \to \infty$, where the exact solution may also be exist. In the next two chapters, we will consider the flow fluid when $Re \gg 1$ or $Re \to 0$.

Chapter4 Very Slow Motion

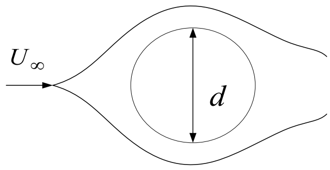
4.1 Equations of motion

Consider a constant density flow, the equations of motion are:

Continuity: $\nabla \cdot \vec{V} = 0$

Momentum: $\rho \left[\dfrac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \right] = -\nabla p + \nabla^2 \vec{V}$

Introduce the
$$\begin{cases} \text{characteristic velocity : } U_\infty \\ \text{characteristic length : } d \\ \text{characteristic pressure : } p_0 \\ \text{characteristic time : } t_0 \end{cases}$$



then the non-dimensional properties become

$$\tilde{\vec{V}} = \frac{\vec{V}}{U_\infty}, \quad \tilde{\vec{r}} = \frac{\vec{r}}{d}, \quad \tilde{p} = \frac{p}{p_0}, \quad \tilde{t} = \frac{t}{t_0}$$

and $\quad \nabla = \dfrac{\partial}{\partial \vec{r}} = \dfrac{1}{d}\dfrac{\partial}{\partial \tilde{\vec{r}}} = \dfrac{\tilde{\nabla}}{d} \Rightarrow \tilde{\nabla} = d\nabla$

(The magnitude of "～" order 1)

Continuity: $\tilde{\nabla} \cdot \tilde{\vec{V}} = 0$

(continuity equation is invarant for non-dimensionalization)

Momentum:

$$\frac{d}{U_\infty t_0}\frac{\partial \tilde{\vec{V}}}{\partial \tilde{t}} + \tilde{\vec{V}} \cdot \tilde{\nabla} \tilde{\vec{V}} = -\frac{P_0}{\rho U_\infty{}^2}\tilde{\nabla}\tilde{p} + \frac{1}{\rho U_\infty d/\mu}\tilde{\nabla}^2 \tilde{\vec{V}}$$

If we denote:

Reynolds No. = $\dfrac{\rho U_\infty d}{\mu}$

And pick up: $P_0 = \rho U_\infty{}^2 =$ dynamic pressure

Equation becomes

$$\underbrace{\underbrace{\frac{d}{U_\infty t_0}\frac{\partial \tilde{\vec{V}}}{\partial \tilde{t}}}_{\text{unsteady part}} + \underbrace{\tilde{\vec{V}} \cdot \tilde{\nabla} \tilde{\vec{V}}}_{\text{convective part}}}_{\text{inertia forces}} = \underbrace{-\tilde{\nabla}\tilde{p}}_{\text{pressure forces}} + \underbrace{\frac{1}{\text{Re}}\tilde{\nabla}^2 \tilde{\vec{V}}}_{\text{viscous forces}}$$

1) If $Re \to \infty \Rightarrow \dfrac{d}{U_\infty t_0}\dfrac{\partial \tilde{\vec{V}}}{\partial \tilde{t}} + \tilde{\vec{V}} \cdot \tilde{\nabla}\tilde{\vec{V}} = -\tilde{\nabla}\,\tilde{p}$ (4.1)

2) If $Re \to 0 \Rightarrow \boxed{\tilde{\nabla}^2 \vec{V} = 0}$ (4.2)

Note that there is no balance term. We want to have a balance term. Multiply (4.11) by Re

$$Re\frac{d}{U_\infty t_0}\frac{\partial \tilde{\vec{V}}}{\partial \tilde{t}} + \underbrace{\cancel{Re\,\tilde{\vec{V}} \cdot \tilde{\nabla}\tilde{\vec{V}}}}_{(0,\ \text{as } Re \to 0)} = -\frac{P_0}{\rho U_\infty^{\ 2}}Re\tilde{\nabla}\,\tilde{p} + \tilde{\nabla}^2\tilde{\vec{V}}$$

(i) The unsteady term coefficient:

For a oscillation body flow, w = frequency of oscillation

we can choose: $t_0 = \dfrac{1}{\omega}$

the first coefficient:

$$Re\frac{d}{U_\infty t_0} = Re\frac{d\omega}{U_\infty} \to 0, \text{ as } Re \to 0 \text{ and } \omega \text{ is not very large}$$

Remark: ① if there is no body oscillation, we may pick $t_0 = U_\infty/d$

② For a highly oscillation body, the unsteady term can't neglected.

(ii) The pressure coefficient

We want to pick up P_0 such that $\dfrac{P_0}{\rho U_\infty^{\ 2}}Re \to 1$, and this term can be left to balance the viscous term. Therefore

$$P_0 = \frac{\rho U_\infty^{\ 2}}{\mathrm{Re}} = \frac{\rho U_\infty^{\ 2}}{\rho U_\infty d/\mu} = \frac{\mu U_\infty}{d}$$

And the momentum equation (4.1) become

$$\boxed{0 = -\tilde{\nabla}\,\tilde{p} + \tilde{\nabla}^2\tilde{\vec{V}}} \qquad (4.3)$$

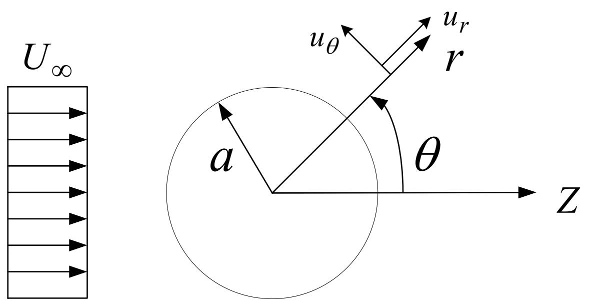
Summary: For a steady, constant density, slow flow ($Re \to 0$)

$$\begin{cases} \tilde{\nabla}\tilde{\vec{V}} = 0 \\ 0 = -\tilde{\nabla}\,\tilde{p} + \tilde{\nabla}^2\tilde{\vec{V}} \end{cases}$$

Also name as: Slow flow, Creeping flow, or stokes´ flow.

4.2 Slow flow past a sphere

consider a steady, constant density flow with $Re \to 0$.



$\vec{V} = 0 \quad \text{on } r = a$

$$\vec{V} = u_r \vec{e}_r + u_\theta \vec{e}_\theta + \cancelto{0}{u_\phi \vec{e}_\phi}$$

[在 upstream 方向上無 ϕ 方向之分量，故在 sphere 附近 u_ϕ 幾乎為零。但 u_θ 和 u_r 則由 $U_\infty \hat{e}_z$ 變化而來，故不可忽略]

Mass: $\quad \operatorname{div}\vec{V} = 0$

$$\Rightarrow \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\underbrace{(r^2 \sin\theta\, u_r)}_{\frac{\partial \psi}{\partial \theta}} + \frac{\partial}{\partial \theta}\underbrace{(r \sin\theta\, u_\theta)}_{-\frac{\partial \psi}{\partial r}}\right] = 0$$

The streamfunction are takes such that the continuity equation is satisfied automatically. Thus

$$r^2 \sin\theta\, u_r = \frac{\partial \psi}{\partial \theta}, \qquad r \sin\theta\, u_\theta = -\frac{\partial \psi}{\partial r}$$

$$\text{or} \quad \boxed{u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}}, \quad \boxed{u_\theta = -\frac{1}{r \sin\theta}\frac{\partial \psi}{\partial r}} \tag{4.4}$$

Momentum equation:

$$0 = -\nabla p - \mu \nabla^2 \vec{V}$$

Since $\quad \nabla^2 \vec{V} = \operatorname{grad}(\cancelto{0}{\operatorname{div}\vec{V}}) - \text{cuel curl}\,\vec{V}$

$$\Rightarrow \boxed{0 = -\nabla p - \mu \,\text{curl curl}\,\vec{V}}$$

take curl on both side

$$0 = 0 - \mu \text{ curl curl } \underbrace{\text{curl}\vec{V}}_{\equiv \vec{\Omega}} \tag{4.5}$$

$$\vec{\Omega} = \text{cuel}\vec{V} \;=\; \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ u_r & ru_\theta & 0 \end{vmatrix}$$

$$= \frac{1}{r^2 \sin\theta}\left\{\hat{e}_r(0) + \hat{e}_\theta(0) + r\sin\theta\,\hat{e}_\phi\left[\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right]\right\}$$

$$= \frac{\hat{e}_\phi}{r}\left\{u_\theta + r\frac{\partial u_\theta}{\partial r} - \frac{\partial u_r}{\partial \theta}\right\} \qquad \text{sub. Eqn (4.4)}$$

$$= \frac{\hat{e}_\phi}{r}\left[-\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} + r\frac{\partial}{\partial r}\left(-\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\right) - \frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right]$$

$$= \frac{\hat{e}_\phi}{r}\left[\cancel{-\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}} - \frac{r}{\sin\theta}\left(\cancel{\frac{-1}{r^2}\frac{\partial\psi}{\partial r}} + \frac{1}{r}\frac{\partial^2\psi}{\partial r^2}\right) - \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right]$$

$$= \frac{-\hat{e}_\phi}{r\sin\theta}\left[\frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right]$$

$$= \Omega\,\hat{e}_\phi$$

Where $\Omega \equiv -\dfrac{1}{r\sin\theta}\mathrm{D}\psi$

$$D \equiv \text{differential operator} \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$

Then

$$\text{Curl }\vec{\Omega} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ 0 & 0 & \Omega \end{vmatrix}$$

$$= \frac{1}{r^2 \sin\theta}\left[\frac{\partial\Omega}{\partial\theta}\hat{e}_r - r\hat{e}_\theta\frac{\partial\Omega}{\partial r}\right]$$

Finally, we can obtain

$$\text{curl curl}\,\vec{\Omega} = \frac{\hat{e}_\phi}{r\sin\theta}\mathrm{D}^2\psi$$

Eq. (4.5) $\Rightarrow$

$$\mathrm{D}^2\psi = 0$$

or

$$\boxed{\left[\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2\psi = 0} \tag{4.6}$$

B.C'S in terms of ψ : (Recall $u_r = \dfrac{1}{r^2\sin\theta}\dfrac{\partial\psi}{\partial\theta}$, $u_\theta = -\dfrac{1}{r\sin\theta}\dfrac{\partial\psi}{\partial r}$)

(i) On $r = a$:

$$\begin{cases} u_r = 0 \rightarrow \dfrac{\partial\psi}{\partial\theta} = 0 \\ u_\theta = 0 \rightarrow \dfrac{\partial\psi}{\partial r} = 0 \end{cases} \tag{4.7a}$$

(ii) Infinity condition

$$\because\ \vec{V}_\infty = U_\infty\hat{e}_z = U_\infty[(\cos\theta)\hat{e}_r + (-\sin\theta)\hat{e}_\theta]$$

$$\therefore\ u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \rightarrow U_\infty\cos\theta \quad \text{as } r\rightarrow\infty \tag{4.8a}$$

$$u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \rightarrow -U_\infty\sin\theta \quad \text{as } r\rightarrow\infty \tag{4.8b}$$

integrate (4.8a) and (4.8b), we obtain

$$\boxed{\psi \sim U_\infty\frac{r^2}{2}\sin^2\theta \quad \text{as } r\rightarrow\infty} \tag{4.7b}$$

Assume: $\psi(\mathrm{r},\theta) = f(\mathrm{r})\sin^2\theta$, then the B.C'S become

(4.7a) $r = a \rightarrow f'(a) = f(a) = 0$

(4.7b) $r\rightarrow\infty \rightarrow f(\mathrm{r}) \sim U_\infty\dfrac{r^2}{2}$ as $r\rightarrow\infty$

Sub. Into Eq. (4.6), we get

$$\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)^2 f = \left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)\left(\frac{d^2 f}{dr^2}-\frac{2f}{r^2}\right)=0$$

Aside: $r^4\frac{d^4 f}{dr^4}+ar^3\frac{d^3 f}{dr^3}+br^2\frac{d^2 f}{dr^2}+cr\frac{df}{dr}+df=0$

we can assume solution to be the form of $f = Ar^n$,

we will have 4 roots for n, n= 1, -1, 2, 4.

$$\therefore \quad f=\frac{A}{r}+Br+Cr^2+Dr^4 \tag{4.9}$$

B.C'S:

(1) $f(r) \sim U_\infty \frac{r^2}{2}$ as $r\to\infty$

compare with (4.9), we observe that we need to take $\boxed{C=\frac{U_\infty}{2}}$ and $\boxed{D=0}$

to satisfy $f(r)\sim U_\infty \frac{r^2}{2}$ for $r\to\infty$. (The value of A, B are not important, since they are not the highest order term, and $r^2 >> r$ as $r\to\infty$)

Eq. (4.9) $\Rightarrow$

$$f=\frac{A}{r}+Br+\frac{U_\infty}{2}r^2 \tag{4.10}$$

$$\left.\begin{array}{l}(2)\ \ f(a)=0 \ \rightarrow\ \frac{A}{a}+Ba+\frac{U_\infty}{2}a^2=0\\ \quad f'(a)=0\ \rightarrow\ -\frac{A}{a^2}+B+U_\infty a=0\end{array}\right\}\Rightarrow$$

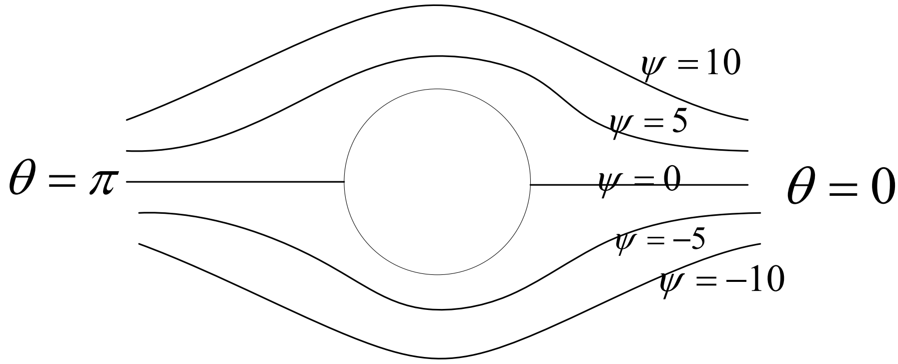
$$A=\frac{1}{4}a^3U_\infty\ ,\ B=-\frac{3}{4}aU_\infty$$

$$\therefore\quad \boxed{\psi(r,\theta)=a^2\,U_\infty\,\sin^2\theta\left[\frac{1}{4}\left(\frac{a}{r}\right)-\frac{3}{4}\left(\frac{r}{a}\right)+\frac{1}{2}\left(\frac{r}{a}\right)^2\right]+\text{const}} \tag{4.11a}$$

$$u_r=U_\infty\cos\theta\left[1-\frac{3}{2}\left(\frac{a}{r}\right)+\frac{1}{2}\left(\frac{a}{r}\right)^3\right]$$

$$u_\theta=-U_\infty\sin\theta\left[1-\frac{3}{4}\left(\frac{a}{r}\right)-\frac{1}{4}\left(\frac{a}{r}\right)^3\right] \tag{4.11 b, c}$$

The streamlines are:



Remark:

(1) The streamlines possess perfect forward – and – backward symmetry: there is no wake. It is the role of the convective acceleration terms, here neglected, to provide the strong flow asymmetry typical of higher Reynolds number flows.

(2) The local velocity is everywhere retarded from its freestream value: there is no faster region such as occurs in potential flow.

(3) The effect of the sphere extent to enormous distance: at $r = 10a$, the velocity are still 10 percent below their freestream values.

(4) The streamlines and velocity are entirely independent of the fluid viscosity.

The pressure distribution is

$$0 = -\nabla p - \mu \,\mathrm{curl}\,\vec{\Omega}$$

or $$\frac{\partial p}{\partial r} = -\mu \frac{1}{r^2 \sin\theta}\frac{\partial \Omega}{\partial \theta} \;;\; \frac{1}{r}\frac{\partial p}{\partial \theta} = -\frac{1}{r\sin\theta}\frac{\partial \Omega}{\partial \theta}$$

integrate the eqns with the known value of Ω, we finally obtain

$$P = P_\infty - \frac{3}{2} a\,\mu\, U_\infty \frac{\cos\theta}{r^2} \tag{4.12}$$

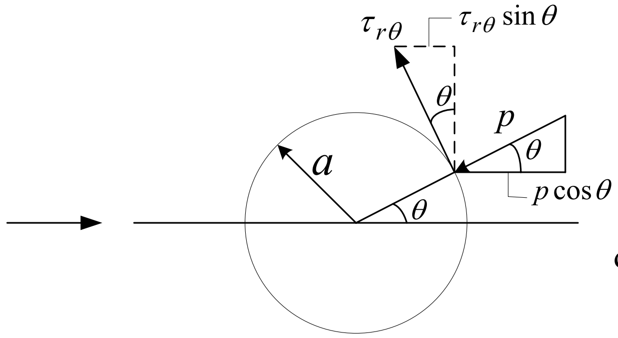
The shear stress in the fluid is

$$\tau_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r}\right) = -\frac{U\mu\sin\theta}{r}\left[1 - \frac{3}{4}\left(\frac{a}{r}\right) + \frac{5}{4}\left(\frac{a}{r}\right)^3\right] \tag{4.13}$$

The drag force on the sphere is thus

$$D = -\int_0^{\pi} \tau_{r\theta}\big|_{r=a} \sin\theta \, \mathrm{dA} - \int_0^{\pi} p\big|_{r=a} \cos\theta \, \mathrm{dA}$$

$$\mathrm{d}A = 2\pi a^2 \sin\theta \, \mathrm{d}\theta$$



$$\mathrm{dA} = 2\pi (\mathrm{a} \sin\theta)(\mathrm{a}\, \mathrm{d}\theta)$$
$$= 2\pi \mathrm{a}^2 \sin\theta \, \mathrm{d}\theta$$

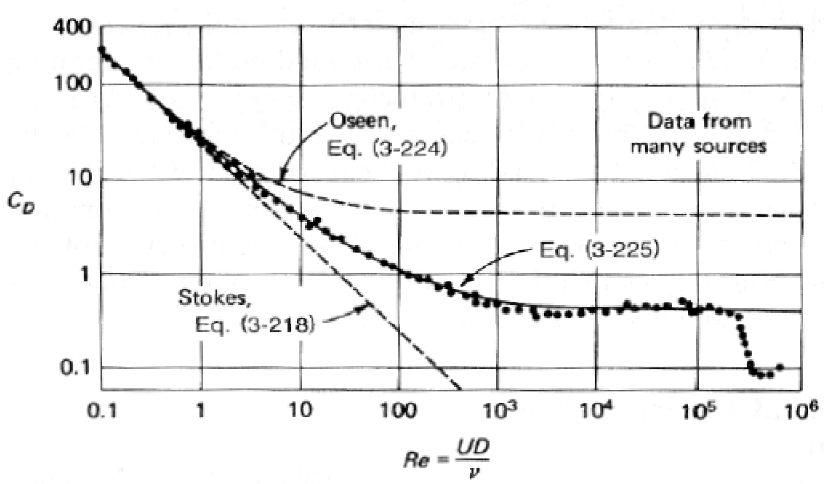
$$\therefore \; D = 3\pi \mu a U_\infty \left[\underbrace{\int_0^{\pi} \sin^3\theta d\theta}_{4/3} + \underbrace{\int_0^{\pi} \cos^2\theta \sin\theta d\theta}_{2/3} \right]$$

$$= \underbrace{4\pi \mu a U_\infty}_{\text{due to friction}} + \underbrace{2\pi \mu a U_\infty}_{\text{due to pressure force}}$$

or $\boxed{D = 6\pi \mu a U_\infty}$ " Stoke's Formula " (4.14)

Define: $\mathrm{Re} = \dfrac{U_\infty 2a}{\nu}$

Then $\boxed{C_D} = \dfrac{D}{\frac{1}{2}\rho U_\infty^{2} (\pi a^2)} = \boxed{\dfrac{24}{\mathrm{Re}}}$ (4.15)



<u>Remarks</u>:

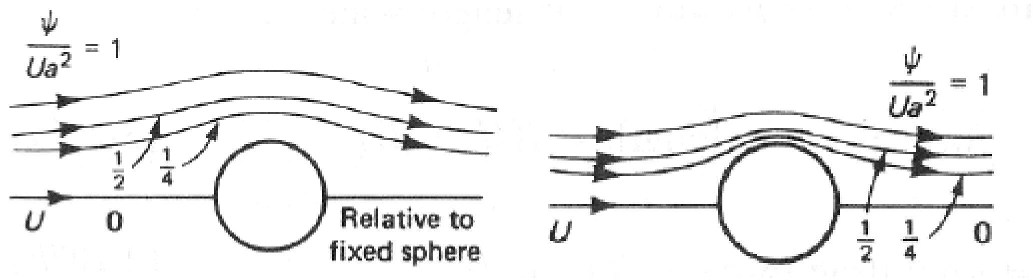
(1) Stokes formula : $D=6\pi\ \mu\, a\, U_\infty$ provides a method to determine the viscosity of a fluid by observing the terminal velocity U_∞ of a small falling ball of radius a.

(2) Stokes formula valid only for Re<1. For Re$\approx$20. These will have separated – flow on the near surface.

(3) For a slow flow, the velocity is not necessarily very small. It could be a very small particle (a<<1) with a high velocity and

$$\mathrm{Re} = \frac{U_\infty a}{\nu} \rightarrow 0.$$

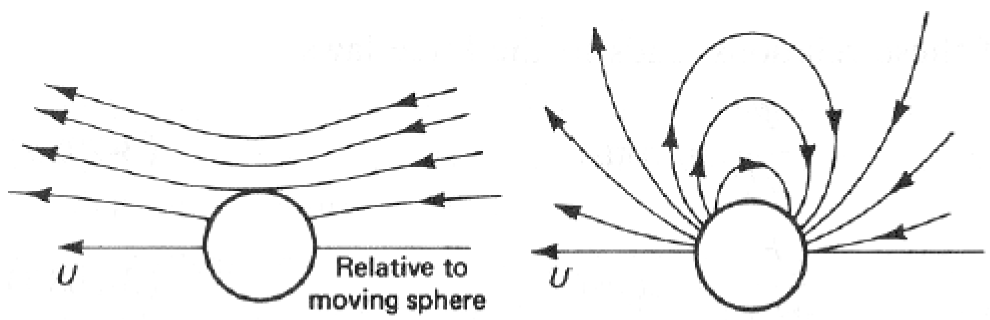
(4) Compare the stokes flow and a potential flow around a fixed sphere:



(Stokes) (Potential)

(Both fore – and – aft symmetric) (Fig. 3-35 White)

The streamline are similar, except that stokes streamlines are displaced further by the body. However, for a sphere moving through a quient fluid.



Stokes flow | Potential flow

Drag the entire surrounding fluid with it

Circulating streamline, indicating that it is merely pushing fluid out of the way

(5) For Re>1. Oseen use perturbation method and obtain a modified formula for C_D.

$$C_D = \frac{24}{Re}(1+\frac{3}{16}Re) \qquad \text{(valid for } Re < 3\sim5) \tag{4.16}$$

Other curve – fitting formula are, for example,

$$C_D \cong \frac{24}{Re} + \frac{6}{1+\sqrt{Re}} + 0.4 \qquad (0 \le Re \le 2\times10^5) \tag{4.17}$$

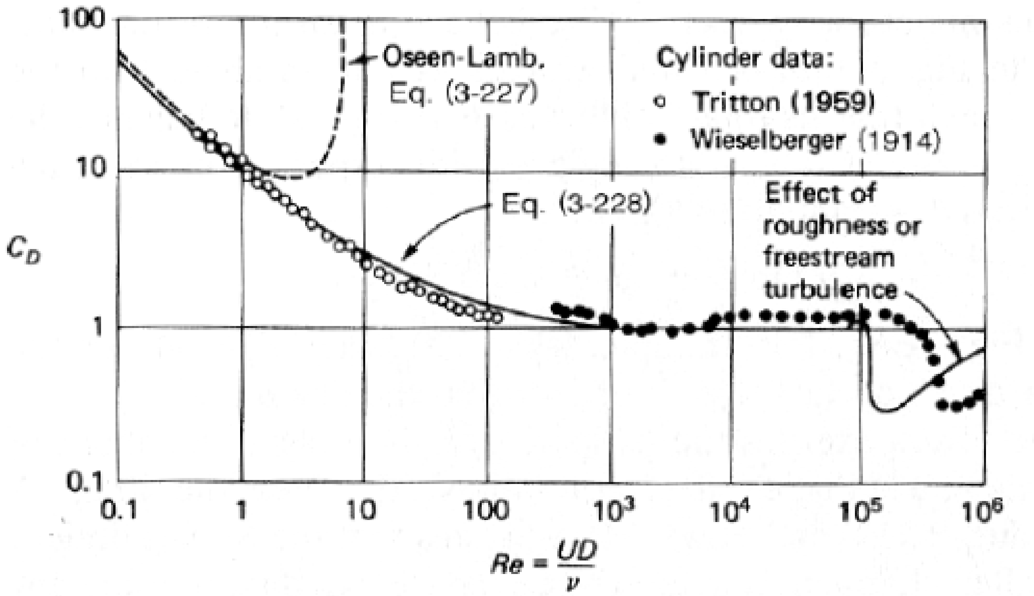
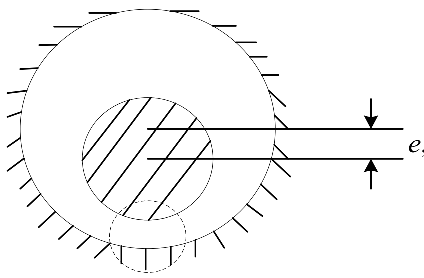


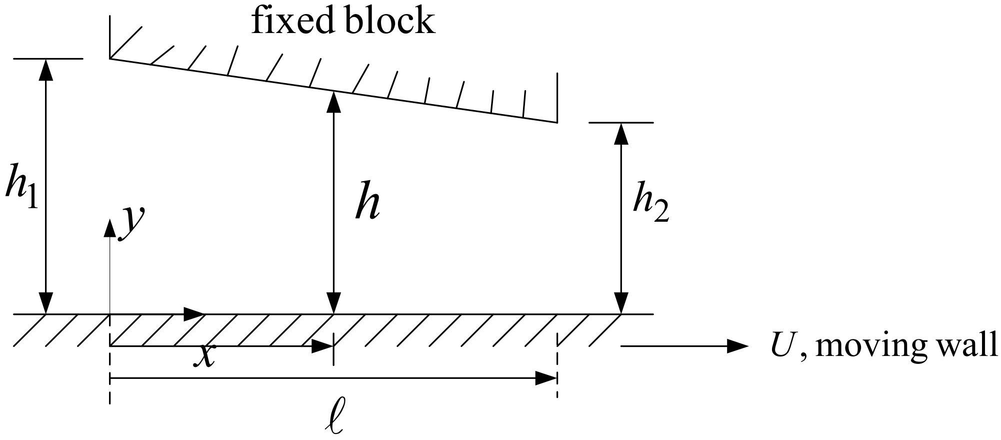
Fig. 3-38 (a) Cylinder data

4.3 The Hydrodynamic Theory of Lubrication (White 3-9.7, p.187-190)

Lubrication between journals and bearings are achieved by filly a thin film of oil between then.



For the sake of simplification, we take a model of



Assume: ① $h << L$

② the sliding surface are very large in z-direction, such that $\partial/\partial z = 0$, $w = 0$

③ steady state

The G..E's become

$$\begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0 \\[2ex] \rho\left(u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}\right)= -\dfrac{\partial p}{\partial x}+\mu\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right) \\[2ex] \rho\left(u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}\right)= -\dfrac{\partial p}{\partial y}+\mu\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right) \end{cases} \quad (4.18)$$

Since $v \ll u$, the y-momentum equation can be totally neglected, that is

$$\frac{\partial p}{\partial y} \cong 0 \qquad \therefore\ p = p(x)$$

The x-momentum, reduces to

$$\rho\, u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \mu \left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} \right)$$

(can be neglected compared with $\frac{\partial^2 u}{\partial y^2}$)

Note that ① $u\frac{\partial u}{\partial x}$ is not zero because the gap width is varied.

② there are two characteristic length h, L in x- and y-directions, thus, the dimensionless parameter must be $\bar{x}$ = x/L, $\bar{y}$=y/h to let the parameters of order 0(1). (Not the same as flow past a sphere where char. Length is diameter d only.)

Compare the order of viscous & inertia forces

$$\frac{\text{Inertia force}}{\text{viscous force}} = \frac{\rho u \frac{\partial u}{\partial x}}{\mu \frac{\partial^2 u}{\partial y^2}} = \left[\frac{\rho \cdot U \cdot \frac{U}{L}}{\mu \cdot \frac{U}{h^2}} \right] = \left[\frac{\rho U L}{\mu} \right] \left[\left(\frac{h}{L} \right)^2 \right]$$

$$\equiv R^* \text{ (reduced Reynolds No.)}$$

<u>Remark</u>:

① $\because$ $h \ll L$, the R^* is generally small even when Re (= $\rho\, UL/\mu$) is large. Thus the Inertia force term can be neglected approximately.

② For example, U = 10 m/s, L = 4 cm

ν = 7×10^{-4} m^2/s, h = 0.1 mm

Re = 570 but R^* = 0.004 only

The x-momentum equation thus becomes

$$\boxed{\frac{dp}{dx} = \mu \frac{\partial^2 u}{\partial y^2}} \qquad (4.19)$$

B.C's:

① $y = 0,\ u = U$

② $y = h,\ u = 0$

③ $x = 0,\ p = p_0$

④ $x = L,\ p = p_0$

This is an assumed assumption for the model. For a certain segment in lubrication fluid, the pressure is not the same on both ends

Note that *dp/dx* here is no longer constant (such as the couette flow between two parallel walls), it must satisfy the pressure P_0 at both ends. The *dp/dx* must be determined in such a way as to satisfy the continuity equation in every section of the form

$$Q = \int (udy + vdx) = \int_0^{h(x)} udy = \text{const} \tag{4.20}$$

The solution of Eq. (4.19) with given B.C's is

$$u = U\left(1 - \frac{y}{h}\right) - \frac{h^2}{2\mu}\left(\frac{dp}{dx}\right)\frac{y}{h}\left(1 - \frac{y}{h}\right) \tag{4.21}$$

Here, *dp/dx* is determined by sub. (4.21) into (4.20), as

$$Q = \frac{Uh}{2} - \frac{h^3}{12\mu}\left(\frac{dp}{dx}\right)$$

Or

$$\frac{dp}{dx} = 12\,\mu\left(\frac{U}{2h^2} - \frac{Q}{h^3}\right) \tag{4.22}$$

integrate with B.C ($p = p_0$ at x=0), we have

$$p = p_0 + 6\,\mu U \underbrace{\int_0^x \frac{dx}{h^2}}_{\equiv\ b_1(x)} - 12\,\mu Q \underbrace{\int_0^x \frac{dx}{h^3}}_{\equiv\ b_2(x)} \tag{4.23}$$

Inserting B.C of $p = p_0$ at $x = L$, we get

$$Q = \frac{1}{2} U \underbrace{\frac{b_1(L)}{b_2(L)}}_{\equiv\ \text{characteristic thickness}\ \equiv H} = \frac{1}{2} UH \tag{4.24}$$

We may conclude the procedure of solution as follows:

(1) Known wedge shape $h(x)$

(2) Obtain $b_1(L)$ & $b_2(L)$, as well as H & Q

(3) The pressure distribution (4.23) can be rewritten as

$$p(x) = p_0 + 6\mu U b_1(x) - 12\mu Q b_2(x) \tag{4.25}$$

and is readily obtained.

(4) The dp/dx, Eq. (4.22) can be written and calculated as

$$\frac{dp}{dx} = \frac{6\mu U}{h^2}(1 - \frac{H}{h}) \tag{4.26}$$

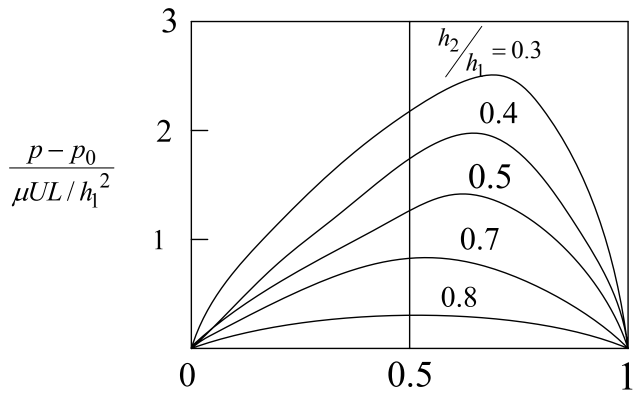
(5) Knowing dp/dx, the velocity distribution can be found from Eq. (4.21)

Remark:

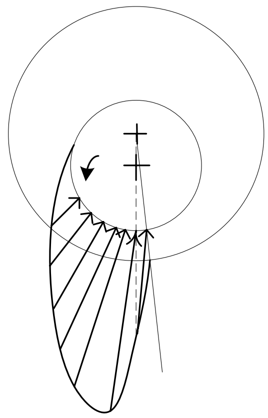
(1) p_{max} or p_{min} occurs where $h = H$.

(2) For a straight wedge with h_1 & h_2 at both ends, we get

$$p(x) = p_0 + 6\mu U \frac{L}{h_1^2 - h_2^2} \frac{(h_1 - h)(h - h_2)}{h^2}$$



as h_2/h_1 decrease the max pressure become higher and occurs near the exit.



For the above example,
with $h_2/h_1 = 0.5$,
the $p_{\max} = 250\,\text{atm}$

(3) Taking from F.M. White text:

”Recall that stokes flow, being linear, are reversible. If we reverse the wall in the figure to the left, that is, U<0, then the pressure change is negative. The fluid will not actually develop a large negative pressure but rather will cavitate and or a vapor void in the gap, as is well shown in the G.I. Taylor film ”Low Reynolds number Hydrodynamics. ”

”Thus flow into an expanding narrow gap may not generally bear much load or provide good lubrication. The effect is unavoidable in a rotating journal bearing, where the gap contracts and then expends, and partial cavitation often occurs. ”

(4) For the case of bearing with finite width in z-direction, it was found that the decrease in thrust supported by such a bearing is very considerable due to the side wide decrease in pressure.

(5) With large U and high temperature (low viscosity), the R^* are nearly or exceeding unity. The result shown above needs to be modified since the inertia term $u\frac{\partial u}{\partial x}$ must be taken into account. As U is too high, turbulent flow may occur.

Chapter5 Boundary Layer Theory

5.1 The Boundary Layer Equations

From the first beginning, we are interested in the phenomena of a flow in high *Re*. In this type of flow, $Re = \dfrac{\text{Inertia force}}{\text{Viscous force}}$ >>1, the inertia force will dominant almost the flow field, except for the region very near the wall, where the effect of viscous force are not negligible.

In order to investigate the governing equation and the thickness of the boundary layer, we use the dimensionless analysis. Introduce the boundary layer thickness δ (not known yet! waiting for being investigated. All the assumption is only $Re >> 1$). And non-dimensionalization

$$u^* = \frac{u}{U}, \quad v^* = \frac{v}{V}, \quad y^* = \frac{y}{\delta}, \quad x^* = \frac{x}{L}, \quad p^* = \frac{-p}{\rho U^2}, \quad t^* = \frac{t}{L/U}$$

so that u^*, v^*, y^*,……etc are all 0(1).

(1) The continuity equation becomes

$$\left(\frac{U}{L}\right)\underbrace{\frac{\partial u^*}{\partial x^*}}_{0(1)} + \left(\frac{V}{\delta}\right)\underbrace{\frac{\partial v^*}{\partial y^*}}_{0(1)} = 0$$

To keep the equation unchanged, it must be

$$\frac{U}{L} \sim \frac{V}{\delta}, \text{ or } \frac{V}{U} \sim 0\left(\frac{\delta}{\ell}\right)$$

i.e. from the continuity equation, we get a relation between *V/U* and δ/L

(2) Sub. Into x-momentum equation of the N-S equation:

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + \underbrace{\frac{V}{U}\frac{\ell}{\delta}}_{①} v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \underbrace{\frac{\nu}{\ell U}}_{②}\frac{\partial^2 u^*}{\partial x^{*2}} + \underbrace{\frac{\nu \ell}{U\delta^2}}_{③}\frac{\partial^2 u^*}{\partial y^{*2}}$$

all the " * " terms are 0(1), we need only to consider the 3 coefficients above.

① $= \dfrac{V}{U}\dfrac{\ell}{\delta}$ ~ 0(1) from continuity equation

② $= \dfrac{\nu}{\ell U} = \dfrac{1}{Re} \rightarrow 0$ ($\because$ Re >>1), therefore, this term can be dropped out compared with other)

③ $\dfrac{\nu\ell}{U\delta^2} =$? In order to keep this term (otherwise, all the viscous term disappear, it becomes the inviscid flow. This is the flow outside the boundary layer, not what we want.) It should be also order of 1. So

$$\frac{\nu\ell}{U\delta^2} \sim 0(1) \rightarrow \delta \sim \sqrt{\frac{\nu\ell}{U}} \sim \sqrt{\frac{\nu x}{U}}$$

that is from dimensionless analysis, we already have a ideal about the boundary layer thickness.

$$\delta \sim \sqrt{\frac{\nu x}{U}}$$

(3) Sub. Into y-momentum equation, we have

$$\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + \underbrace{(\frac{V}{U})(\frac{\ell}{\delta})}_{①} v^*\frac{\partial v^*}{\partial y^*} = -\underbrace{\frac{U}{V}\frac{\ell}{\delta}}_{②}\frac{\partial p^*}{\partial y^*} + \underbrace{\frac{\nu}{\ell U}}_{③}\frac{\partial^2 u^*}{\partial x^{*2}} + \underbrace{\frac{\nu\ell}{U\delta^2}}_{④}\frac{\partial^2 u^*}{\partial y^{*2}}$$

where ① $= \dfrac{V}{U}\dfrac{\ell}{\delta}$ ~ 0(1) (continuity equation) (o.k.)

③ $= \dfrac{\nu}{\ell U} = \dfrac{1}{Re} \rightarrow 0$ ($\because$ Re >>1) can be neglected

④ $\dfrac{\nu\ell}{U\delta^2} = \dfrac{1}{Re}(\dfrac{\ell}{\delta})^2$ ~ 0(1) from the result of x-momentum equation

② $-\dfrac{U}{V}\dfrac{\ell}{\delta}$ >> 0(1) , so that we can see that this term are larger than other terms, the *y*-momentum equation can be written contained only dominant term as $\partial p / \partial y = 0$

i.e. we can conclude $p = p(x)$ only.

So we conclude:

For a flow with Re >>1, the flow very near the wall is governed by the equation (incompressible flow)

$$\begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0 \\ \dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{dp}{dx}+\nu\dfrac{\partial^2 u}{\partial y^2} \end{cases} \tag{5.1}$$

and $\dfrac{\partial p}{\partial y}=0$. This equation is called"boundary layer equation"

Remark:

Compare the Navier-stokes equation and the boundary layer equation, and explain why the latter is easier to be solved numerically?

(Ans:) For simplicity, let's consider the incompressible flow as an example:

Navier-stokes equation:

$$\begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0 \\ \dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{dp}{dx}+\nu\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right) \\ \dfrac{\partial v}{\partial t}+u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}+\nu\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right) \end{cases}$$

Boundary layer equation:

$$\begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0 \\ \dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+\nu\dfrac{\partial^2 u}{\partial y^2} \\ \dfrac{\partial p}{\partial y}=0 \end{cases}$$

There are many things to be noticed:

(1) Continuity equation is not affected by the consideration of Reynolds number.

(2) $p=p_e(x)$ in Boundary-layer equation, and is determined by the Bernoulli equation outside the boundary layer..

$$\frac{dp_e}{dx} = -\rho U_e \frac{dU_e}{dx} \tag{5.2}$$

where x is the coordinate parallel to the wall.

(3) The equation becomes parabolic in B-L theory, with x as the marching variable. In computer, parabolic equation is easier to solve than the elliptic equation, which the N-S equation belongs to.

(4) Boundary conditions:

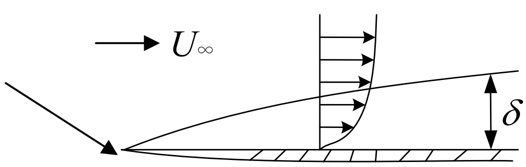
In B-L equation

(i) $\frac{\partial^2 v}{\partial y^2}$, $\frac{\partial v}{\partial x}$, $\frac{\partial^2 v}{\partial x^2}$ have been discarded, only $\frac{\partial v}{\partial y}$ left. Therefore, we need only one boundary condition of v on y-direction. The obvious condition to retain is no slip: $v = 0$ at y=o.

(ii) $\frac{\partial^2 u}{\partial x^2}$ has been discarded. Therefore, one condition of u in x-direction (to satisfy $\frac{\partial u}{\partial x}$) is sufficient. The best choice is normally the inlet plane, and the u in the exit plane will yield the correct value without our specifying them.

(iii) Boundary condition of u on y has no change. There are two conditions to satisfy $\frac{\partial^2 u}{\partial y^2}$.

Namely $\begin{cases} u=0 & \text{as } y = 0 \\ \frac{\partial u}{\partial y}=0 & \text{as } y = \delta \end{cases}$



Steady state

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\[2ex] u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\dfrac{\partial^2 u}{\partial y^2} \\[2ex] 0 = -\dfrac{\partial p}{\partial y} \rightarrow P = P_e(x) \end{cases}$$

B.C's

$u(x, y = 0) = 0$

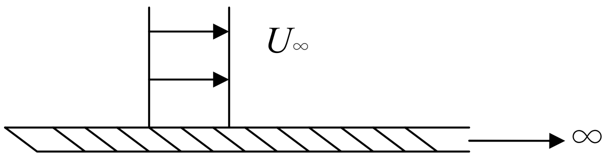
$v\,(x, y = 0) = 0$

$u(x, y = \delta\,) = U_e \leftarrow$ This condition must match the inner limit of the outer (inviscid) flow

$v(x, y = \infty) = U_e$

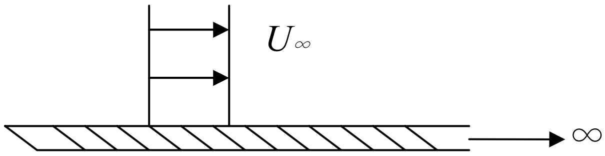
(not the same as U_∞)

inviscid



5.2 Flat plate case (infinite far)

inviscid



$U_e = U_\infty$ = constant; $p_e = p_\infty$ = constant

Known the inviscid properties, we next go to the boundary layer problem.

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 0 \\[2ex] u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = \nu \dfrac{\partial^2 u}{\partial y^2} \end{cases}$$

: $y = 0$ $\begin{cases} u = 0 \\ v = 0 \end{cases}$

$y = \infty$ $u = u_\infty$ $\rightarrow$(since here, we stand in the Boundary layer. We can see only B.C, so the edge of the B.C seen ∞ for me.)

Introduce stream function

$$u = \frac{\partial \psi}{\partial y},\ v = -\frac{\partial \psi}{\partial x}$$

the continuity equation can be satisfied automatically. Sub into momentum equation , we can get one depended variable $\rightarrow$ mess equation. (hard to be solved)

Similarity solution:

Introduce (try)

$$\eta = \frac{y}{x}\left(\frac{U_\infty x}{\nu}\right)^\alpha$$

find α, so that a single variable differential equation is obtained in terms of η only.

Can we determine a similarity variable

$$\eta = \eta(x, y), \quad u = \tilde{u}(\eta)$$

so that we can reduce a PDE $\rightarrow$ ODE. Assume

$$\eta = \frac{y}{x}\left(\frac{U_\infty x}{\nu}\right)^{\alpha} \rightarrow \alpha = 1/2$$

$$\left[\begin{array}{l} \text{Dimension of length:} \quad x, y, \dfrac{\nu}{U_\infty} \\ \text{Dimensionless of length:} \quad \tilde{y} = \dfrac{y}{\frac{\nu}{U_\infty}} = \dfrac{U_\infty y}{\nu}, \ \tilde{x} = \dfrac{U_\infty x}{\nu} \end{array}\right]$$

$$\frac{U}{U_\infty} = fn(\eta) = f'(\eta)$$

[anticipate $f(\eta)$ as a dimensionless stream function.]

$$\frac{\partial \psi}{\partial y} = U_\infty f'(\eta)$$

$$\frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = U_\infty f'(\eta)$$

$$\frac{\partial \psi}{\partial \mu}\left(\frac{U_\infty}{\nu}\right)^{1/2}\frac{1}{x^{1/2}} = U_\infty f'(\eta)$$

$$\frac{\partial \psi}{\partial \mu} = U_\infty\left(\frac{\nu}{U_\infty}\right)^{1/2} x^{1/2} f'(\eta)$$

$$\psi = U_\infty\left(\frac{\nu}{U_\infty}\right)^{1/2} x^{1/2} f(\eta) + \text{const}$$

$\eta = 0$ so that $\psi = 0$ represent the body shape($\because$ along body ($y=0$)$\rightarrow \eta = 0$, but $f(0)=0$ from B.C.)

$$\therefore \left\{\begin{array}{l} \eta = \dfrac{y}{x}\left(\dfrac{U_\infty x}{\nu}\right)^{\alpha} \\ \psi = U_\infty\left(\dfrac{\nu}{U_\infty}\right)^{1/2} x^{1/2} f(\eta) \end{array}\right.$$

$$u = \frac{\partial \psi}{\partial y} = U_\infty f'(\eta)$$

$$v = -\frac{\partial \psi}{\partial x} = -U_\infty\left(\frac{\nu}{U_\infty}\right)^{1/2}\left[\frac{1}{2x^{1/2}} f(\eta) + x^{1/2}\frac{df}{d\eta}\frac{\partial \eta}{\partial x}\right]$$

$$\frac{\partial \eta}{\partial x} = \left(\frac{\nu}{U_\infty}\right)^{1/2}\left[-\frac{1}{2x}\frac{y}{x^{1/2}}\right] = -\frac{\eta}{2x}$$

$$v = -U_\infty\left(\frac{\nu}{U_\infty}\right)^{1/2}\left[\frac{f}{2x^{1/2}} - \frac{\eta f'}{2x^{1/2}}\right]$$

$$\boxed{v = -U_\infty\left(\frac{\nu}{U_\infty}\right)^{1/2}\frac{1}{2x^{1/2}}[f - \eta f']}$$

Sub. Into equation, we finally get

$$2f''' + ff'' = 0$$

B.C.
$$\begin{cases} f'(0) = 0 & \left(\frac{u}{U_\infty} = f'(\eta) = 0 \text{ at } y = 0 \text{ or } \eta = 0\right) \\ f(0) = 0 & \left(\frac{v}{U_\infty} = 0 \quad \text{at } \eta = 0\right) \\ f'(\infty) = 1 & (y \to \infty, \quad u = U_\infty) \end{cases} \tag{5.3}$$

This is called"<u>Blasius Problem</u>"

<u>Blasius Equation:</u>

$$\eta = \frac{y}{x}\sqrt{\frac{U_\infty x}{\nu}} = \frac{y}{\sqrt{\nu x / U_\infty}}$$

$$\psi = U_\infty\left(\frac{\nu x}{U_\infty}\right)^{1/2} f(\eta)$$

$$\frac{u}{U_\infty} = f'(\eta), \qquad \frac{\nu}{U_\infty} = \frac{1}{2}\sqrt{\frac{\nu}{U_\infty x}}(\eta f' - f)$$

Sub. into the momentum equation: $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$

$$\begin{cases} \frac{\partial(u/U_\infty)}{\partial x} = \frac{d}{d\eta}\left(\frac{u}{U_\infty}\right)\frac{d\eta}{dx} = f''\left(\frac{-\frac{1}{2}y}{x\sqrt{\nu x / U_\infty}}\right) = -f''\frac{\eta}{2x} \\ \frac{\partial(u/U_\infty)}{\partial y} = \frac{d}{d\eta}\left(\frac{u}{U_\infty}\right)\frac{d\eta}{dy} = f''\left(\frac{1}{\sqrt{\nu x / U_\infty}}\right) \\ \frac{\partial^2(u/U_\infty)}{\partial y^2} = f'''\left(\frac{1}{\nu x / U_\infty}\right) \end{cases}$$

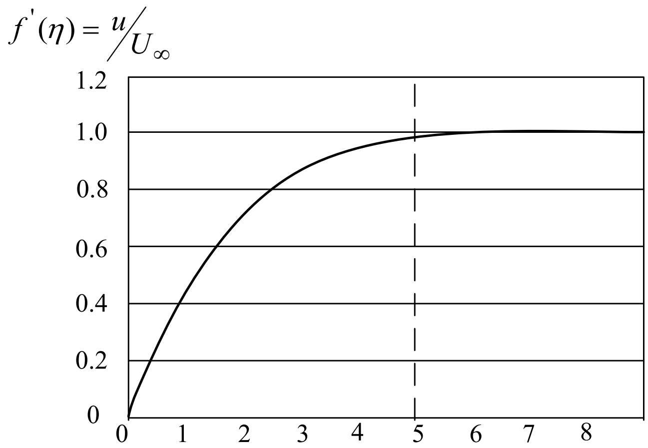
$\Rightarrow$ $\boxed{2f''' + ff'' = 0}$ "Blasius equation"

B.C'S: ① $f(0)=0$

② $f'(0)=0$

③ $f'(\infty)=0$

> Solve Blasius equation by series express
>
> $f = A_0 + A_1\eta + A_2\dfrac{\eta^2}{2} + \ldots.$
>
> → or using Runge-Kutta numerical method to solve it.



$\eta = y \Big/ \sqrt{\dfrac{\nu x}{U_\infty}}$ (p.136)

Fig.7.7 or Table 7.4 on p.139 of H. Schlichting

(or Table 4.1 Fig. 4 - 6 on p.236 of white)

Boundary Layer thickness:

Engineering Argumemt: $y=\delta$ when u/U_∞ =0.99 from Blasius table, we find $f'(\eta)$ =0.99 when η=5.

$$\therefore\ 5 = \eta = \frac{y}{\sqrt{\dfrac{\nu x}{U_\infty}}} = \frac{\delta}{\sqrt{\dfrac{\nu x}{U_\infty}}}$$

$$\therefore \frac{\delta}{x} = \frac{5}{\sqrt{\dfrac{U_\infty x}{\nu}}} = \frac{5}{\sqrt{\mathrm{Re}_x}}$$

or $$\boxed{\delta = 5\sqrt{\frac{\nu x}{U_\infty}}} \qquad (5.4)$$

Surface friction:

$$\tau = \mu \frac{\partial u}{\partial y}$$

$$\tau_w = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$\because \ u = U_\infty f'(\eta)$$

$$\frac{\partial u}{\partial y} = U_\infty f''(\eta)\frac{\partial \eta}{\partial y} = U_\infty f''(\eta)\frac{1}{\sqrt{\nu x / U_\infty}}$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{U_\infty{}^2 f''(0)}{\sqrt{\frac{\nu x}{U_\infty}}}$$

$$\therefore \boxed{\tau_w = \frac{\mu U_\infty{}^2 f''(0)}{\sqrt{\nu U_\infty x}}}$$

$$\boxed{C_f} = \frac{\tau_w}{\frac{1}{2}\rho U_\infty{}^2} = \frac{2f''(0)}{\sqrt{\frac{U_\infty x}{\nu}}} = \boxed{\frac{2f''(0)}{\sqrt{\text{Re}_x}}}$$

Since

$$f''(0) = 0.332$$

$$\therefore \quad \boxed{C_f = \frac{0.644}{\sqrt{\text{Re}_x}}} \tag{5.5a}$$

$$\tau_w = 0.332\, \mu\, U_\infty \left(\frac{U_\infty}{\nu x}\right)^{1/2} \tag{5.5b}$$

The Drag on the flat plate is

$$D = \int_0^L \tau_w(x)dx \qquad \text{(for unit depth)}$$

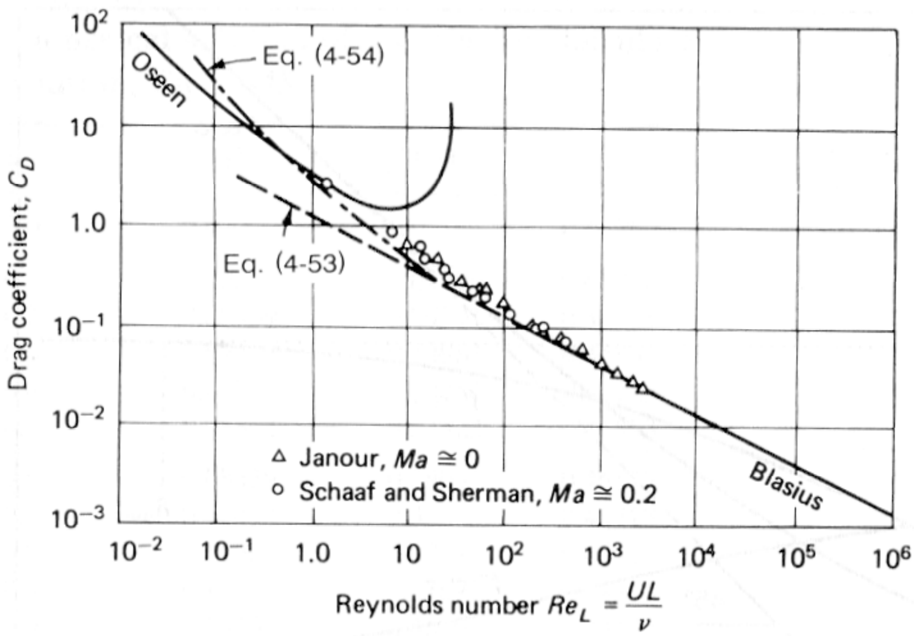
$$= 0.644\, U_\infty \sqrt{\mu \rho L U_\infty} \tag{5.6a}$$

And for a plate wetted on both side

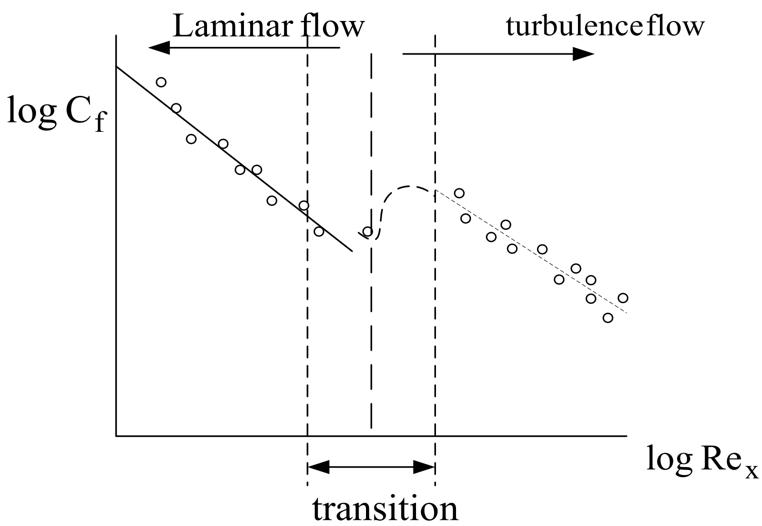
$$D' = 2D = 1.328\, U_\infty \sqrt{\mu \rho L U_\infty} \tag{5.6b}$$

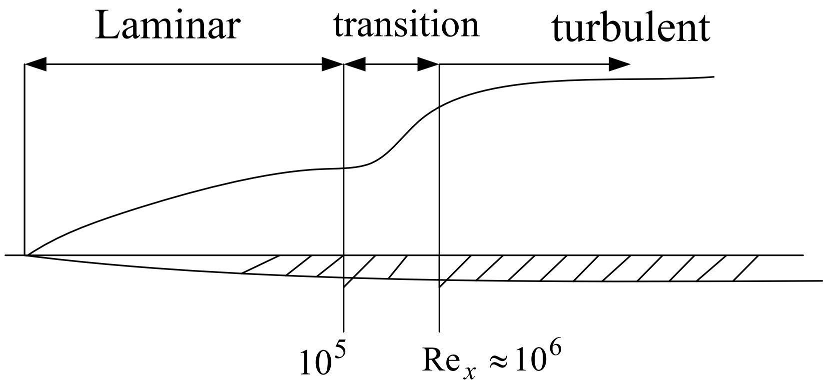
As Remarked on White book, the boundary-layer approximation is not realized until $Re \geq 1000$. For Re_L (= UL/ν) ≤ 1, the Oseen theory is valid. In the Range of $1 < Re_L < 1000$, the correction C_D is given as

$$C_D \approx \frac{1.328}{\sqrt{\mathrm{Re}_L}} + \frac{2.3}{\sqrt{\mathrm{Re}_L}}$$



(p.238 of White)





The displacement thickness δ_1 is defined as

$$\delta_1 = \int_{y=0}^{\infty} (1-\frac{u}{U_\infty})dy$$

$$= \sqrt{\frac{\nu x}{U_\infty}}\int_{\eta=0}^{\infty} [1-f'(\eta)]d\eta$$

$$= \sqrt{\frac{\nu x}{U_\infty}}[\eta_1 - f(\eta_1)]$$ where η_1 denotes a point outside the B.L (η_1>5)

Take $\eta_1 = 7, \quad f(7) = 5.27926$

$\eta_1 = 8, \quad f(8) = 6.27923$ $\Bigg\} \rightarrow \eta_1 - f(\eta_1) \cong 1.7208$

Therefore

$$\boxed{\delta_1 = 1.7208\sqrt{\frac{\nu x}{U_\infty}}} \qquad \text{(displacement thickness)} \qquad (5.7)$$

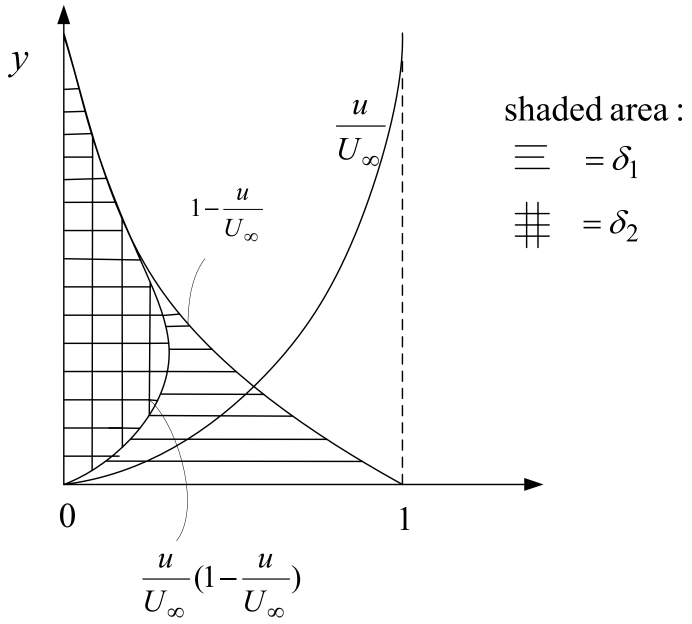
.The momentum thickness δ_2 is defined as

$$\delta_2 = \int_0^{\infty} \frac{u}{U_\infty}(1-\frac{u}{U_\infty})dy$$

$$= \sqrt{\frac{\nu x}{U_\infty}}\int_0^{\infty} f'(1-f')\,d\eta$$

or

$$\boxed{\delta_2 = 0.664\sqrt{\frac{\nu x}{U_\infty}}} \tag{5.8}$$



5.3 Similarity Solutions

For the B.L. equation with pressure gradient. i.e.

$$\begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0 \\[2ex] u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}=U_e\dfrac{dU_e}{dx}+\nu\dfrac{\partial^2 u}{\partial y^2} \end{cases}$$

Do we always have similarity solution? (P.D.E $\rightarrow$ O.D.E)

Nondimensionalized by:

$$U=\frac{u}{U_\infty},\qquad V=\frac{v\sqrt{\text{Re}}}{U_\infty},\qquad U_e=\frac{u_e}{U_\infty}$$

$$X=\frac{x}{L},\qquad Y=\frac{y\sqrt{\text{Re}}}{L},\qquad Re=\frac{U_\infty L}{\nu}$$

Then the equations become

$$\begin{cases} \dfrac{\partial U}{\partial X}+\dfrac{\partial V}{\partial Y}=0 \\[2ex] U\dfrac{\partial U}{\partial X}+V\dfrac{\partial U}{\partial Y}=U_e\dfrac{dU_e}{dX}+\dfrac{\partial^2 U}{\partial Y^2} \end{cases} \tag{5.9}$$

With B.C's

$$U(X,0)=0,\quad V(Y,0)=0,\quad U(X,\ \infty)=U_e(X)$$

The continuity equation is satisfied by the introducing of stream function

$$U=\frac{\partial\psi}{\partial Y},\qquad V=-\frac{\partial\psi}{\partial X}$$

And also introduce

$$\begin{cases} \eta=\dfrac{Y}{g(X)} \\[2ex] \zeta=X \end{cases}$$

$\leftarrow$ g(X) is what we want to find to get the similarity solution.

C.f. for the zero-pressure gradient flow (Blasius Flow)

$$\eta=\frac{y}{x}\sqrt{\frac{U_\infty x}{\nu}}=\frac{Y}{\sqrt{X}}$$

That is we transform the coordinate system $(X, Y) \rightarrow (\zeta, \eta)$.

(Note: later, we will let the variables depend only on η, but not ζ, such that the Non-dimensional velocity profile is independent of the ζ (or X), and the solution is then call "similar " solution.)

And $$\frac{U}{U_e} = \frac{\partial f(\varsigma,\eta)}{\partial \eta}$$ (Later, we hope $f(\zeta, \eta) \rightarrow f(\eta)$!)

$$= \frac{1}{U_e} \frac{\partial \psi(\varsigma,\eta)}{\partial Y} = \frac{1}{U_e} \frac{\partial \psi}{\partial \eta} \frac{\partial \eta}{\partial Y} = \frac{1}{U_e} \frac{\partial \psi}{\partial \eta} \frac{1}{g(\varsigma)}$$

$\rightarrow$ $$\frac{\partial \psi}{\partial \eta} = U_e(\zeta)\, g(\zeta) \frac{\partial f}{\partial \eta}$$

or $$\boxed{\psi(\zeta, \eta) = U_e(\zeta)\, g(\zeta) f(\zeta, \eta)} \qquad (5.10)$$

$$U(\zeta, \eta) = \frac{\partial \psi}{\partial Y} = \frac{\partial \psi}{\partial \eta} \frac{\partial \eta}{\partial Y} = U_e(\zeta)\, g(\zeta) \frac{\partial f}{\partial \eta} \cdot \frac{1}{g(\varsigma)} = U_e(\zeta) \frac{\partial f}{\partial \eta}$$

$$V(\zeta, \eta) = -\frac{\partial \psi}{\partial X} = -\left\{ \frac{d}{d\varsigma}(U_e g) f + (U_e g) \left[\frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial X} + \frac{\partial f}{\partial \varsigma} \underbrace{\frac{\partial \varsigma}{\partial X}}_{=1} \right] \right\}$$

$$\left(\frac{\partial \eta}{\partial X} = -\frac{Y g'}{g^2} = -\frac{Y}{g} \frac{g'}{g} = -\eta \frac{g'}{g} \right)$$

$$g' = \frac{dg(\varsigma)}{d\varsigma}$$

$$= -\left\{ (U_e g)' f + (U_e g) \left[-\eta \frac{g'}{g} \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial \varsigma} \right] \right\}$$

Why we define $u/U_\infty = f(\eta)$, but at sometimes we define $u/U_\infty = f'(\eta)$?

(sol): It's a matter of convenience only,. If we want to use stream function ψ, since $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$ in Cartesian coordinate, thus , we would define u as $u/U_\infty = f'(\eta)$ such that ψ can be expressed as function of $f(\eta)$. Otherwise, ψ must be expressed as an integral form, which is not convenient to use.

Sub. into Eq. (5.9), we obtain

$$f_{\eta\eta\eta} + \alpha(\zeta) f f_{\eta\eta} + \beta(\zeta)(1 - f_\eta^{\,2}) = g^2 U_e (f_\eta f_{\eta\zeta} - f_\zeta f_{\eta\eta}) \qquad (5.11)$$

Where $\begin{cases} \alpha(\zeta) = g(U_e g)' \\ \beta(\zeta) = g^2 U_e' \end{cases}$

We hope to reduce the equation to be a function of η only, also f be a function of η only. Therefore, we pick up

$$\begin{cases} f = f(\eta) \text{ only} \\ \alpha = \text{const} \\ \beta = \text{const} \end{cases} \qquad (5.12)$$

Eqn (5.11) then becomes

$$\boxed{f''' + \alpha f f'' + \beta(1 - f'^2) = 0} \qquad (5.13)$$

(Note: that Blasius equation is a special case of this with α=1, β=0)

B.C's: (1) $f_\eta(0) = f(0) = 0$

(2) $f_\eta(\infty) = 1$

Question: What are the condition for $U_e(\zeta)$ and $g(\zeta)$ under which α and β are retained constant?

Ans: That is, we didn't know $U_e(\zeta)$ and $g(\zeta)$ yet, and we try to express them in terms of constants α and β.

$$\alpha = g(U_e g)' = g^2 U_e' + g g' U_e$$

$$\alpha - \beta = g g' U_e$$

and

$$2\alpha - \beta = 2 g^2 U_e' + g g' U_e = (g^2 U_e)'$$

integrate once

$$g^2 U_e = (2\alpha-\beta)\zeta + C \qquad (\because \alpha, \beta \text{ are const.}, \ \therefore 2\alpha-\beta = \text{const.})$$

$$\alpha-\beta = g g' U_e = g g' \{\frac{1}{g^2}[(2\alpha-\beta)\zeta + C]\}$$

$$= \frac{g'}{g}[(2\alpha-\beta)\zeta + C]$$

or

$$\frac{dg}{g} = \frac{(\alpha-\beta)d\varsigma}{(2\alpha-\beta)\varsigma + C}$$

$$\ln g = \frac{(\alpha-\beta)}{2\alpha-\beta}\ln[(2\alpha-\beta)\zeta + C] + \underbrace{\text{const.}}_{\equiv -\ln k} \quad (2\alpha-\beta \neq 0) \tag{5.14}$$

$$\rightarrow \qquad kg = \left[(2\alpha-\beta)\varsigma + C\right]^{\frac{\alpha-\beta}{2\alpha-\beta}}$$

let $k = 1/k_0$

$$\left.\begin{array}{ll} \rightarrow & \boxed{g = k_0\left[(2\alpha-\beta)\varsigma + C\right]^{\frac{\alpha-\beta}{2\alpha-\beta}}} \\ \text{and} & \boxed{U_e = \frac{1}{k_0^{\,2}}\left[(2\alpha-\beta)\varsigma + C\right]^{\frac{\beta}{2\alpha-\beta}}} \end{array}\right\} \tag{5.15}$$

Let C=0, α=1, k_0=1 and define

$$\text{m} = \frac{\beta}{2\alpha-\beta} \quad \text{or} \quad \beta = \frac{2\alpha\, m}{1+m} = \frac{2m}{1+m}$$

then Eq. (5.15) →

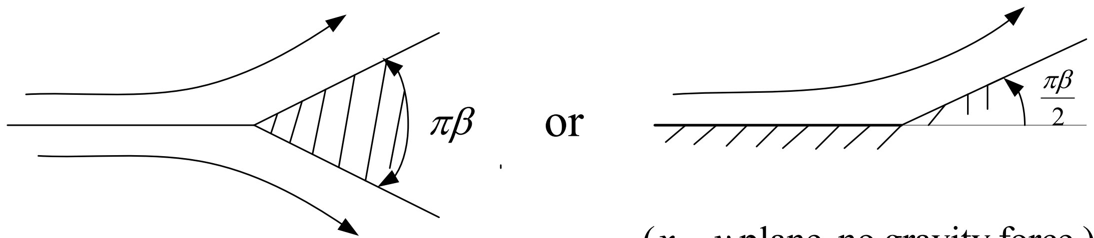
$$U_e = \underbrace{\frac{1}{k_0^{\,2}}\left(\frac{2}{1+m}\right)^{m}}_{\equiv U_0}\zeta^{m}$$

or $\boxed{U_e = U_0\zeta^{m}}$ (5.16a)

$$g = \left(\frac{2\varsigma}{1+m}\right)^{1/2} U_e^{-1/2} \tag{5.16b}$$

$$\eta = \frac{Y}{g} = \frac{YU_e}{\sqrt{\frac{2\varsigma}{1+m}}} \tag{5.16c}$$

This is called the" Falkner-Skan Problem". From potential flow theory, the Eq. (5.16b) is corresponding to an inviscid flow passing a wedge of angle $\pi\beta$.



($x - y$ plane, no gravity force.)

Special cases:

(1) For $m = 1 \rightarrow \beta = 1$, stagnation flow

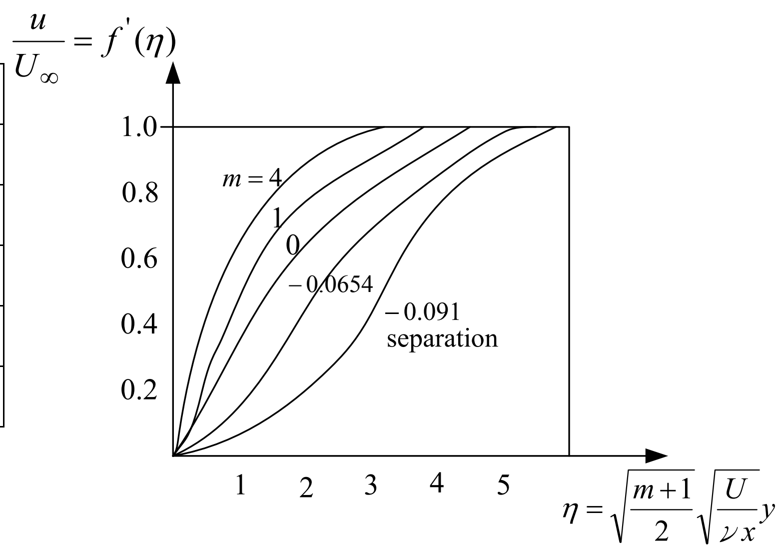
(2) For $m = 0 \rightarrow \beta = 0$, flat plate at zero incidence.

The solution of Eq.(5.13) namely

$$f''' + ff'' + \beta(1 - f'^2) = 0$$

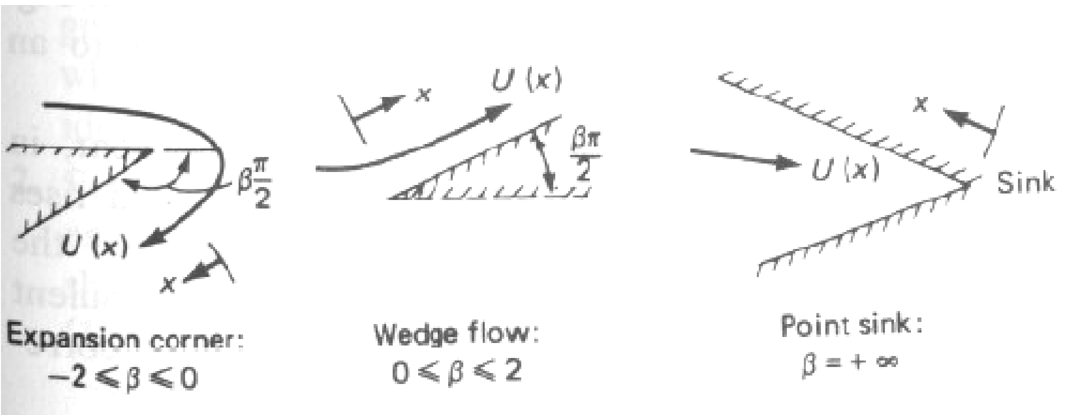
with $\quad f(0) = f'(0) = 0, \quad f'(1) = 1$ is

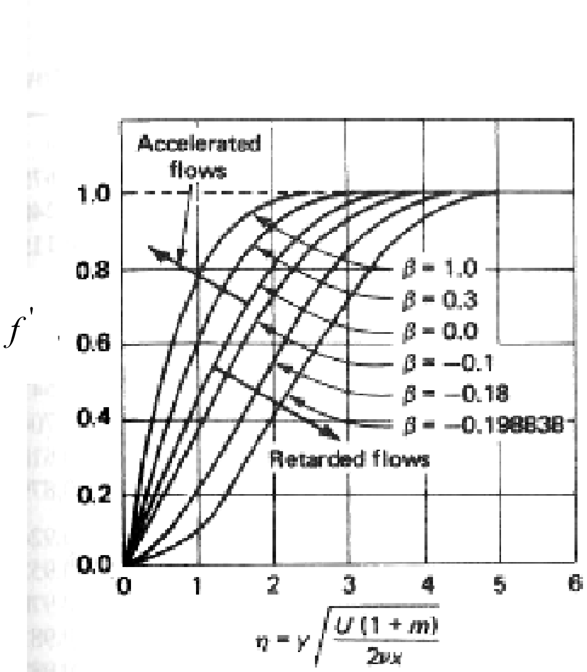
m	$\pi\beta$ $\rightarrow$		β
-0.091	-0.199π	-35.8°	-0.199
-0.0654	-0.14π	-25.2°	-0.14
0	0	0°	0
1	π	180°	
4	1.6π	288°	

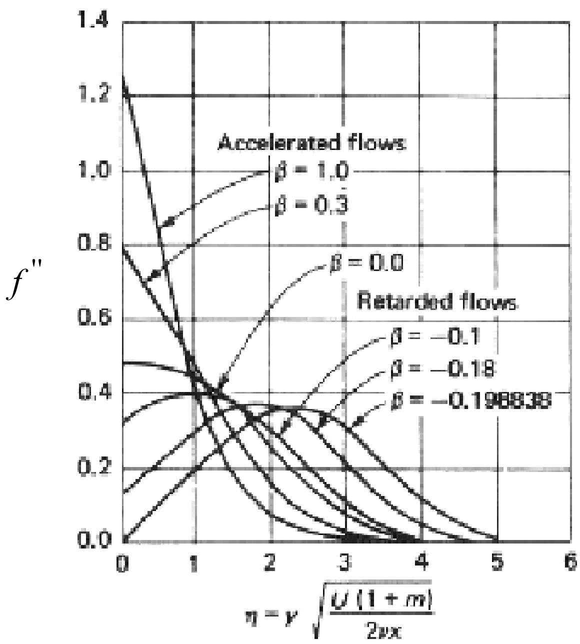


Types of Falkner-Skan flow:

	β	m	Corresponding flow
(1)	$-2 \le \beta \le 0$	$-1/2 \le m \le 0$	Flow around an expansion corner of turning angle $\pi\beta/2$
(2)	0	0	Flat plate
(3)	$0 \le \beta \le 2$	$0 \le m \le \infty$	Flow against a wedge of half-angle $\pi\beta/2$
	(B=1	m=1	Plane stagnation flow, wedge of 180°)
(4)	4	-2	Doublet flow near a plane wall
(5)	5	-5/3	Doublet flow near a 90° corner
(6)	$+\infty$	-1	Flow toward a point sink







Note:

(1) For incompressible wedge flow. As the inclined angle $\pi\beta/2$ increased, the fluid is accelerated, and the boundary layer becomes thinner. However, the τ_w is increased.

Remark:

(1) From $f^{'} \sim \eta$ figure, we can see that boundary layer grows thicker & thicker as β decreasing. (For β = -0.199, separation occurs at y=0.)

(2) From $f^{''} \sim \eta$ figure:

$f^{''}$ corresponding to shearing stress. For β>0, the shearing stress decreases as η increases. However, as $\beta < 0$, the $f^{''}$ rises and they decrease as η increases. This is because

$$\frac{dp_e}{dx} = \mu \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} = \left.\frac{\partial \tau}{\partial y}\right|_{wall}$$

Thus for $\beta < 0$ (decelerating flow, $\frac{dp_e}{dx} > 0$). The $\left.\frac{\partial \tau}{\partial y}\right|_{wall} > 0$, therefore,

$f^{''}$ will rise near wall as η increases.

(3) From $f^{''} \sim \eta$ figure:

$-0.199 \le \beta \le 0$ ← there are (at least) two solution

$\beta < -0.199$ ←multiple solution

(See F.M. White. P.245 for detail)

(4) As the N-S equations are no unique, the B.L. equations also show multiple solutions.

(5) As described in Dr. Sepri's Note, the conditions leading to a similar solution are:

(i) B.C need to be similar → $(\rho v)_0$ restricted in form

(ii) I.C is similar, that is, can't accept an arbitrary $f_0(\eta)$

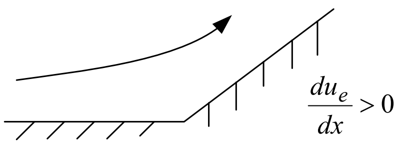
(iii) External pressure gradient must comply with β = const.

(iv) Density profile is similar.

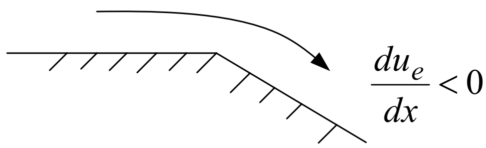
As m = -0.091, $\left.\dfrac{\partial u}{\partial y}\right|_{y=0} = U_\infty f_{\eta\eta}\Big|_{\eta=0} = 0$, therefore, the separation occurs. We conclude that

$$
\begin{cases}
\text{If } m > 0 \\
\quad \dfrac{dU_e}{dX} > 0, \Rightarrow \dfrac{dp_e}{dx} = -\rho U_e \dfrac{dU_e}{dX} < 0 \\
\quad\quad \Rightarrow \text{accelerating flow} \\
\text{If } m < 0 \text{ (but } -1/2 < m) \\
\quad \dfrac{dU_e}{dX} < 0, \quad \Rightarrow \dfrac{dp_e}{dx} > 0 \Rightarrow \text{decelerating flow}
\end{cases}
$$

In this course, the flow is taken as <u>incompressible</u>; therefore, the flow is a accelerated as it past a wedge and decelerated as it past a corner.

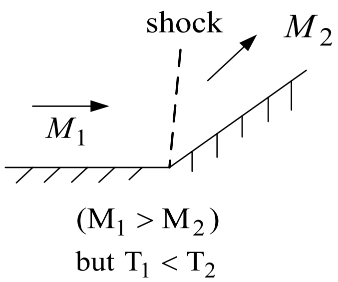


(subsonic nozzle)

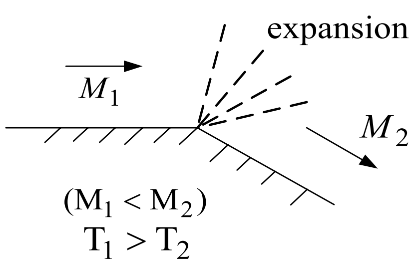


(subsonic diffuser)

However, as the flow is <u>compressible</u>, it will be different, e.g.



(supersonic diffser)



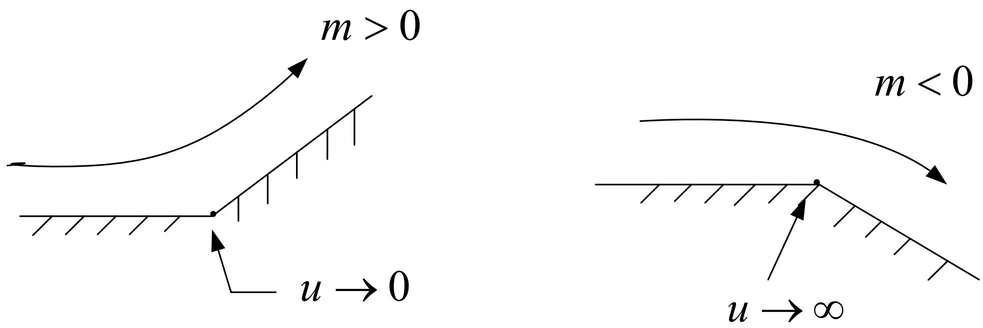
(supersonic nozzle)

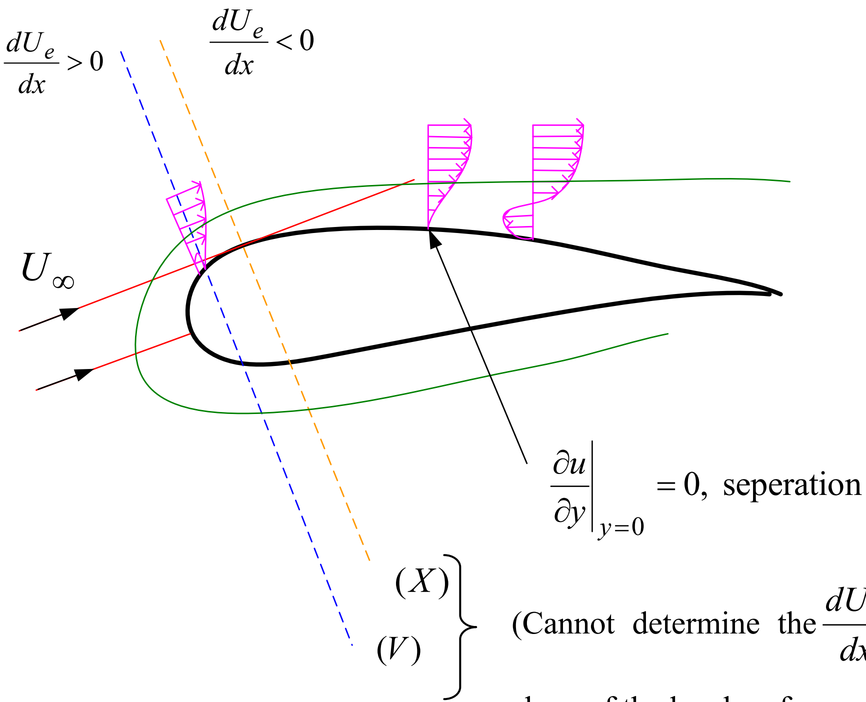
Since $M = \dfrac{V}{\sqrt{\gamma RT}} \rightarrow V = M\sqrt{\gamma RT}$

It hard to tell whether $V_1 > V_2$ or $V_1 < V_2$

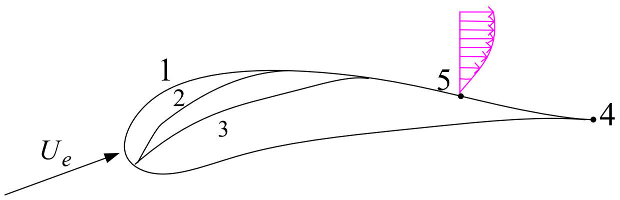
But normally $V_1 < V_2$

(In x-y plane, no gravity force acting)





(Cannot determine the $\frac{dU_e}{dx}$>0 or $\frac{dU_e}{dx}$<0 from the slope of the local surface w.r.t the free-stream direction. It normally further upstream as shown)



For curve 1-4, 2-4 & 3-4, the velocity profile at 5 will not be the same.

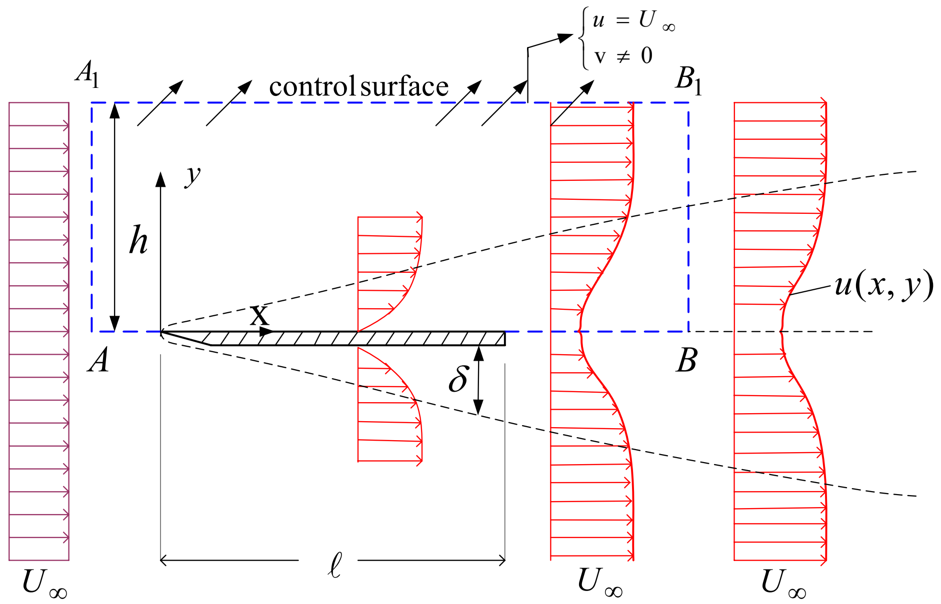
Problem:

Show that (δ^*/τ_w) dp/dx represents the ratio of pressure force to wall friction force in the fluid in a boundary layer. Show that it is constant for any of the Falkner-Skan wedge flows. (J. schetz. P. 92, prob. 4.6)

5.4 Flow in the wake of Flat Plate at zero incidence

Preface: the B.L. equation can be applied not only in the region near a solid wall, but also in a region where the influence of friction is dominating exists in the interior of a fluid. Such a case occurs when two layers of fluid with different velocities meet, such as: **wake and jet**.

Consider the flow in the wake of a flat plate at zero incidences



Want to find out: (1) the velocity profile in the wake.

Assume: $dp/dx = 0$

For the mass flow rate: ($\Sigma = 0$)

At AA_1 section $= \rho\int_0^h U_\infty dy$ (entering)

At BB_1 section $= -\rho\int_0^h u dy$ (leaving)

At AB section $= 0$

At A_1B_1 section $= -\rho\int_0^h (U_\infty - u)dy$ ← (To keep $\Sigma_{mass} = 0$)

Actually along A_1–B_1, the $u = U_\infty$, the mass must be more out to satisfy continuity $\dot{m} = -\rho\int_A^B v(x, h)dx$

For the x-momentum floe rate:

At AA_1 section $= \rho \int_0^h U_\infty^2 \mathrm{d}y$ (entering)

At BB_1 section $= -\rho \int_0^h u^2 \mathrm{d}y$ (leaving)

At AB section $= 0$

At A_1B_1 section $= \dot{m}_{AB}\, U_\infty = U_\infty \left[-\rho \int_0^h (U_\infty - u)\mathrm{d}y\right] = -\rho \int_0^h U_\infty (U_\infty - u)\mathrm{d}y$

Drag on the upper surface = Σ Rate of change of x-momentum in A_1-B_1-B-A

$$= \rho \int_0^h u(U_\infty - u)\mathrm{d}y \tag{5.17}$$

In order to calculate the velocity profile, let us first assume a velocity defect $u_1(x, y)$ as

$$u_1(x, y) = U_\infty - u(\mathrm{x}, \mathrm{y}) \tag{5.18}$$

and $u_1 \ll U_\infty$, which occurs some distance downstream of the trailing edge of the plate ($\mathrm{x} > 3\ell$). Substituting (5.18) into the B.L. equation, namely

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

gives

$$(U_\infty - u_1)\frac{\partial}{\partial x}(\cancel{U_\infty} - u_1) + v\frac{\partial}{\partial y}(\cancel{U_\infty} - u_1) = \nu \frac{\partial^2}{\partial y^2}(\cancel{U_\infty} - u_1)$$

after neglecting the **high order terms** of u_1, it yields

$$\boxed{U_\infty \frac{\partial u_1}{\partial x} = \nu \frac{\partial^2 u_1}{\partial y^2}} \tag{5.19}$$

$\left(\underset{\ll 1}{u_1}\ \underset{\ll 1}{\frac{\partial u_1}{\partial x}},\quad \underset{\ll 1}{v}\ \underset{\ll 1}{\frac{\partial u_1}{\partial x}}\right)$

we can neglect the h.o.T. of u_1, since $u_1 \ll U_\infty$

With B.C's:

(i) $y = 0, \quad \dfrac{\partial u_1}{\partial y} = 0$ (5.20a)

(ii) $y \to \infty, \quad u_1 = 0$ (5.20b)

In order to transform the P.D.E. to a O.D.E., we introduce a new variable similar to the Blasius method for the flat plate as

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} \tag{5.21}$$

and assume that

$$u_1 = C\, U_\infty\, f(\eta)(\ell/x)^{1/2} \tag{5.22}$$

Aside: the reason for taking $x^{-1/2}$ in u_1 is that

$$D = \rho\int_0^h u(U_\infty - u)dy \approx \rho\int_0^h U_\infty\, u_1 dy \approx \rho\int_0^h U_\infty\, u_1\left(\frac{\nu x}{U_\infty}\right)^{1/2} dy$$

To make D independent of x so that the solution is similar along x-direction, u_1 must $\sim x^{-1/2}$

Substituting Eq.(5.21) & (5.22) into (5.19) gives

$$\frac{d^2 f}{d\eta^2} + \frac{1}{2}\eta\frac{df}{d\eta} + \frac{1}{2}f = 0 \tag{5.23}$$

with B.C's

(i) $\eta = 0$, $\dfrac{\partial f}{\partial \eta} = 0$ (5.24a)

(ii) $\eta \to \infty$, $f \to 0$ (5.24b)

Integrate once

$$\frac{df}{d\eta} + \underbrace{\frac{1}{2}\int_0^\eta \eta\frac{df}{d\eta}d\eta}_{(\int u\,dv = uv - \int v\,du)} + \cancel{\frac{1}{2}\int_0^\eta f d\eta} = C_1$$

cancel

$$\left(= \frac{1}{2}\eta f - \cancel{\frac{1}{2}\int_0^\eta f(\eta)d\eta}\right)$$

$$\Rightarrow \quad \boxed{\frac{df}{d\eta} + \frac{1}{2}\eta f(\eta) = C_1 = 0} \tag{5.25}$$

From (5.24a)

$$\ln f = -\frac{\eta^2}{4} + C_2 \qquad \therefore\ f = C_3 e^{-\eta^2/4}$$

Without lose of generality, we can set C_3=1, and therefore

$$f(\eta) = e^{-\eta^2/4} \tag{5.26}$$

Sub. (5.26) back to Eq. (5.22) to get

$$\boxed{u_1 = C\, U_\infty\, (\ell/x)^{1/2}\, e^{-\eta^2/4}} \tag{5.27}$$

and

$$D = \rho U_\infty^2\, C\left(\frac{\nu\ell}{U_\infty}\right)^{1/2} \int_0^\infty f(\eta)\, d\eta \qquad \left(\because \int_0^\infty e^{-\eta^2/4}\, d\eta = \pi^{1/2}\right)$$

$$= \rho U_\infty^2\, C\sqrt{\pi}\, \sqrt{\frac{\nu\ell}{U_\infty}}$$

Compare with the exact solution which we have obtained as before as

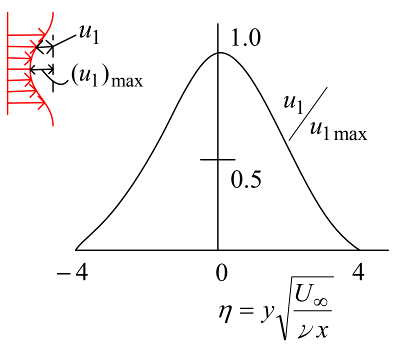
$$D = 0.664\, \rho U_\infty^2 \left(\frac{\nu\ell}{U_\infty}\right)^{1/2} \quad \text{(-one-side flat plate)} \tag{5.6a}$$

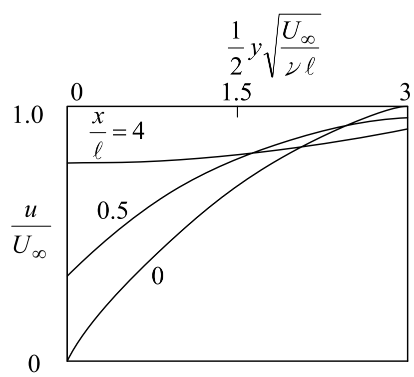
We can get $\quad C = \dfrac{0.664}{\sqrt{\pi}}$

Therefore

$$\boxed{u_1 = \underbrace{\frac{0.664}{\sqrt{\pi}}\, U_\infty \left(\frac{\ell}{x}\right)^{1/2}}_{\text{(Amplitude)}}\ \underbrace{e^{-y^2 U_\infty / 4\nu x}}_{\text{(decaying factor)}}} \tag{5.28}$$

The velocity distribution is:





(p.179 in Schlichting)

Remark:

(1) Eq.(5.19) is a linear conduction equation, so it can actually be by separation variables easily.

(2) A"wake" is the"defect" in stream velocity behind an immersed body in a flow.

(3) A slender plane body with zero lift produces a smooth wake whose velocity defects u_1 decays monotonically downstream.

(4) A blunt body, such as a cylinder, has a wake distorted by an alternating shed vortex structure.

(5) A lifting body will superimpose shed vortices of one sense.

(6) From velocity profile, we can assume boundary edge ($u_1/U_{max} \approx 0.01$) occurs when $\eta \sim 4$ thus

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}}$$

$$\rightarrow \quad 4 = \delta\sqrt{\frac{U_\infty}{\nu x}}$$

$$\therefore \delta = \frac{4}{\sqrt{U_\infty / \nu x}} = \frac{4}{(U_\infty \nu)}\sqrt{\frac{U_\infty x}{\nu}}$$

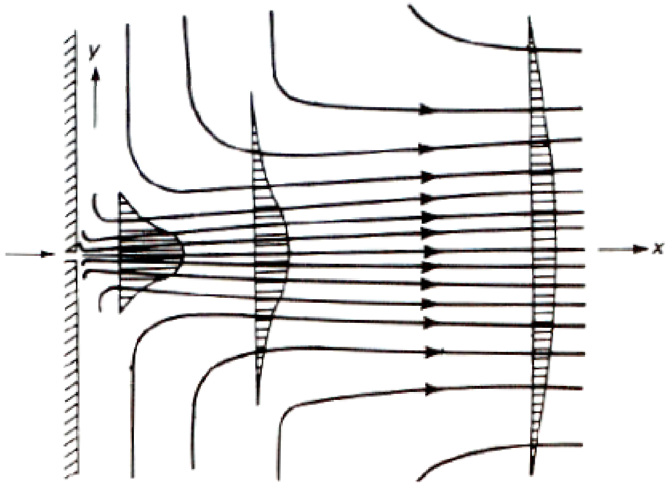
$\therefore \delta \sim Re_x^{1/2}$ (similar to the B.L. thickness growing in upper surface of a flat plate)

(See p.22 of Van-Dyke book)

(7) In meet cases, the wake flow becomes turbulent due to the stability of the wake flow. From velocity profile, there is a part of inflexion, which will cause the unstability of the flow structure.

5.5 Two-Dimensional Laminar Jet

Consider a 2-D Laminar Jet



$$\frac{dp}{dx} = 0$$

The total momentum of the Jet remains constant, i.e., independent of the x, or

$$J = \rho \int_{-\infty}^{\infty} u^2 \, \mathrm{d}y = \text{const} \tag{5.29}$$

Assume

$$u \sim f'\left(\frac{y}{x^q}\right) \tag{5.30}$$

and the stream function

$$\psi \sim x^p f\left(\frac{y}{x^q}\right) = x^p f(\eta), \text{ where } \eta = \frac{y}{x^q} \tag{5.31}$$

We now need to determine p & q.

(i) J = constant

$$\Rightarrow \int_{-\infty}^{\infty} \left(\frac{\partial \psi}{\partial y}\right)^2 \mathrm{d}y = \text{independent of } x.$$

$$\Rightarrow \int_{-\infty}^{\infty} \left[x^p \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y}\right]^2 x^q \, \mathrm{d}\eta = \int_{-\infty}^{\infty} \left[x^{p-q} \frac{\partial f}{\partial \eta}\right]^2 x^q \, \mathrm{d}\eta = \text{independent of } x.$$

$\Rightarrow$ power of x: $(p - q) \times 2 + q = 0$

$$\Rightarrow \boxed{2p - q = 0} \tag{5.32}$$

(ii) From momentum equation:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \underbrace{(p-q)+(p-q-1)}=p-3q$$

$$\Rightarrow \boxed{p+q=1} \tag{5.33}$$

From (5.32) & (5.33): p = 1/3, q = 2/3

Therefore

$$\eta \sim \frac{y}{x^{2/3}} \quad \Rightarrow \quad \eta = C_2\frac{y}{x^{2/3}} \tag{5.34}$$

$$\psi \sim x^{1/3} f(\frac{c_2 y}{x^{2/3}}) \Rightarrow \quad \psi = C_1 x^{1/3} f(\frac{c_2 y}{x^{2/3}}) \tag{5.35}$$

Thus $\quad u=\dfrac{\partial\psi}{\partial y}=C_1C_2\, x^{-1/3}\dfrac{df}{d\eta}$

$$v = -\frac{\partial\psi}{\partial x}= \frac{1}{3}C_1\, x^{-2/3} f(\eta)+C_1\, x^{1/3}(-2/3)\frac{c_2 y}{x^{5/3}}\frac{df(\eta)}{d\eta}$$

$$= \frac{1}{3}\ x^{-2/3}\ C_1\,[f(\eta)-x^{-2/3}\,2\eta\frac{df}{d\eta}]$$

Sub. Into the momentum equation, we can get

$C_1/3 = \nu C_2$ (such that the terms contain x, $f(x)$ are omitted and the

P.D.E→ O.D.E)

Choose $C_1 = \nu^{1/2}$, $C_2 = 1/(3\nu^{1/2})$

$$\eta = \frac{y}{3\nu^{1/2}x^{2/3}} \tag{5.36a}$$

$$\psi = \nu^{1/2}\, x^{1/3} f(\eta) \tag{5.36b}$$

$$u = \frac{1}{3}x^{-1/3}\frac{df}{d\eta} \tag{5.36c}$$

$$v = \frac{1}{3}x^{-2/3}\,\nu^{1/2}\,[f(\eta)-2\eta\frac{df}{d\eta}] \tag{5.36d}$$

Substituting (5.36c) & (5.36d) into the momentum equation, we obtain

$$\boxed{\underbrace{(\frac{df}{d\eta})^2}_{①} + \underbrace{f(\frac{d^2 f}{d\eta^2})}_{②} + \underbrace{\frac{d^3 f}{d\eta^3}}_{③} = 0} \tag{5.37}$$

With B.C's:

$$\begin{cases} \left.\dfrac{\partial u}{\partial y}\right|_{y=0} = 0 \quad \rightarrow \quad \left.\dfrac{d^2 f}{d\eta^2}\right|_{\eta=0} = 0 \\ v|_{y=0} = 0 \quad \rightarrow \quad f(0) = 0 \\ u|_{y=0} \rightarrow 0 \quad \rightarrow \quad \dfrac{df}{d\eta} \rightarrow 0 \text{ as } \eta \rightarrow \infty \end{cases} \tag{5.38}$$

Integrate Eq. (5.37) by past:

$$① = \int_0^\eta (\frac{df}{d\eta})^2 d\eta = \int_0^\eta \left\{ \frac{d}{d\eta}[f \frac{df}{d\eta}] - f \frac{d^2 f}{d\eta^2} \right\} d\eta$$

$$= \int_0^\eta \frac{d}{d\eta}[f \frac{df}{d\eta}] d\eta - \int_0^\eta f \frac{d^2 f}{d\eta^2} d\eta$$

$$= f(\frac{df}{d\eta}) - \int_0^\eta f \frac{d^2 f}{d\eta^2} d\eta$$

$$② = \int \frac{d^3 f}{d\eta^3} d\eta = \frac{d^2 f}{d\eta^2}$$

So that Eq. (5.8) becomes

$$\boxed{\frac{d^2 f}{d\eta^2} + f(\frac{df}{d\eta}) = C_1 = 0} \tag{5.39}$$

$(\because f''(0) = f(0) = 0, \ \therefore \ C_1 = 0)$

Define: $\zeta = a\eta$

$f(\eta) = 2a\, F(\zeta)$

$$\Rightarrow \quad \frac{df}{d\eta} = 2a \frac{dF}{d\varsigma} a, \quad \frac{d^2 f}{d\eta^2} = 2a \frac{d^2 F}{d\varsigma^2} a^2 \tag{5.40a,b}$$

Sub. Into Eq. (5.39):

$$\frac{d^2 F}{d\varsigma^2} + 2F \frac{dF}{d\varsigma} = 0$$

integrate once

$$\frac{dF}{d\varsigma} + F^2 = C_2 = 1$$

(Let C_2 =1 without loss of generality)

(Since we haven't determine the value of"a", thus, we can set C_2 equal to arbitrary value without ref. to the B.C.)(If we do not set $\zeta = a\eta$, $f(\eta) = 2a\,F(\zeta)$, then the integration C_2 cannot be arbitrary, we need to determine coefficient C_2 by keep J = const.)

$$\Rightarrow \int \frac{dF/d\varsigma}{1-F^2} = \int 1\, d\varsigma$$

$$\Rightarrow \tanh^{-1} F = \zeta + \cancel{C_3}$$

0 ($\zeta = 0, f = F = 0, \ \therefore C_3 = 0$)

$$\Rightarrow \tanh \zeta = F$$

From (5.40a)

$$\frac{dF}{d\zeta} = 2a^2\,(1 - \tanh^2 \zeta)$$

$$\Rightarrow \boxed{u = \frac{2a^2}{3x^{1/3}}(1 - \tanh^2 \zeta)} \tag{5.41}$$

The constant"a" is remained to determine. We can get "a" from the J value which is a known value.

$$J = \rho \int_{\infty}^{-\infty} u^2\, dy = \frac{2x^{2/3}\nu^{1/2}3}{a} \ \frac{4}{9}\frac{\rho a^4}{x^{2/3}} \underbrace{\int_0^{+\infty} (1 - \tanh^2 \zeta)^2\, d\zeta}_{=2/3}$$

$$= \frac{16}{9}\nu^{1/2} a^3 \rho$$

Therefore

$$\boxed{a = \left(\frac{9}{16}\frac{J}{\rho\nu^{1/2}}\right)^{1/3}} \tag{5.42}$$

and

$$u_{\max} = u\big|_{y=0} = 0.45\left(\frac{J^2}{\rho\mu x}\right)^{1/3} \tag{5.43}$$

The volume rate of discharge across any vertical plane is

$$\dot{Q} = \int_{-\infty}^{\infty} u\,dy = 3.3019\left(\frac{J\nu x}{\rho}\right)^{1/3} \tag{5.44}$$

or $\qquad \dot{m} = \rho\dot{Q} = 3.3019(J\rho\mu x)^{1/3}$

From Eq.(5.43) we know that the max axial velocity decreases as x increases. However, from Eq.(5.44) we downstream direction, because fluid particles are carried away with the jet owing to friction on its boundary. It also increases with increasing momentum.

<u>Remark:</u>

(1) Note that $\dot{m} \sim x^{1/3}$ because the jet entrains ambient fluid by dragging it along. However, Eq.(5.44) implies falsely that $\dot{m}=0$, which is the slot where the Re $\sim \dfrac{\dot{m}}{\mu} \sim \left(\dfrac{J\rho x}{\mu^2}\right)^{1/3}$. The B.L. theory is not valid for Re is small.

Therefore, we cannot ascertain any details of the flow near the jet outlet with B.L. theory.

(2) Since jet velocity profile are S-Shaped (i.e. have a point of inflection), they are unstable and undergo transition to turbulent early – at a Re of about 30, based on exist slot width and mean slot velocity.

(3) Define the width of the jet as twice the distance y where $u = 0.01\ u_{\max}$, we then have

$$\text{Width} = 2\,y\big|_{u=1\%\,u_{\max}} \approx 2.18\left(\frac{x^2\mu^2}{J\rho}\right)^{1/3}$$

Thus $\boxed{\text{Width} \sim x^{2/3}}$ and $\boxed{\sim J^{-1/3}}$

Chapter 6 Approximate methods for the Solution of the 2-D, steady B.L. Equations

In the history of the developing the B.L. flow, we have:

(1) Analysis solution (exact solution): A exact solution consists every term in the B.L. equation although some of terms may be identically zero. We do not imply that an exact solution is one in a closed form; it could be a convergent series.

→ For as complex geometry (specially with pressure gradient), this method is difficult and sometimes impossible. We have discussed some simple case in the previous chapter.

(2) Approximate Solutions: All approximate methods are integral methods which do not attempt to satisfy the B.L. equation for every streamline; instead, the equations are satisfied only on an average extended over the thickness of the B.L. → well-suited to the generation of a quick outline of a solution even in more complex cases. This technique is important before the advent of fast computer.

6.1 Karman's Integral Momentum Relation

Consider a steady, 2-D, compressible flow:

Continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

$$\frac{(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{6.1}$$

B.L. Eq:

$$\rho \left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right] = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left[\mu \frac{\partial u}{\partial y} \right]$$

Since
$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \left[\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} \right] + \left[\frac{(\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} \right] \cdot u$$

$$= \left[\rho u \frac{\partial u}{\partial x} + u \frac{(\rho u)}{\partial x} \right] + \left[\rho v \frac{\partial u}{\partial y} + u \frac{\partial (\rho v)}{\partial y} \right]$$

$$= \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho u v)$$

$$\therefore \quad \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho u v) = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left[\mu \frac{\partial u}{\partial y} \right] \tag{6.2}$$

Integrate the continuity equation from $y = 0$ to $y = \delta$:

$$\underbrace{\int_0^\delta \frac{(\rho u)}{\partial x} \, dy}_{} + \underbrace{\int_0^\delta \frac{\partial (\rho v)}{\partial y} \, dy}_{} = 0$$

(Leibnitz's rule)

$$\overbrace{\frac{\partial}{\partial x}\int_0^\delta \rho u \, dy - \rho_e u_e \frac{d\delta}{dx}} + \rho_e u_e - \rho_0 u_0 = 0$$

$$\Rightarrow \quad v_e = \frac{1}{\rho_e}\left[-\frac{\partial}{\partial x}\int_0^\delta \rho u \, dy + \rho_e u_e \frac{d\delta}{dx} + \rho_0 u_0 \right] \tag{6.3}$$

Integrate the B.L. equation:

$$\underbrace{\int_0^\delta \frac{(\rho u^2)}{\partial x} \, dy}_{①} + \underbrace{\int_0^\delta \frac{\partial (\rho u v)}{\partial y} \, dy}_{②} = \underbrace{\int_0^\delta \left(-\frac{dp}{dx}\right) dy}_{③} + \underbrace{\int_0^\delta \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) dy}_{④}$$

$$① \;=\; \frac{\partial}{\partial x}\int_0^{\delta} \rho\, u^2\, \mathrm{d}y - \rho_{\mathrm{e}} u_{\mathrm{e}}^{2} \frac{d\delta}{dx} \qquad \text{(Leibnitz's Rule)}$$

$$② \;=\; \rho_{\mathrm{e}} u_{\mathrm{e}} v_{\mathrm{e}} - \rho_0 \underbrace{\cancel{u_0}}_{(=0,\ \because\ u_0 = 0)} v_0 = \rho_{\mathrm{e}} u_{\mathrm{e}} v_{\mathrm{e}} \underbrace{=}_{\text{(Eq.(6.3))}} u_{\mathrm{e}} \left[-\frac{\partial}{\partial x}\int_0^{\delta} \rho\, u\, \mathrm{d}y + \rho_{\mathrm{e}} u_{\mathrm{e}} \frac{d\delta}{dx} + \rho_0 u_0\right]$$

$$③ \;=\; -\left(\frac{dp}{dx}\right)\delta \qquad \left(\because \frac{dp}{dx} = fn(x) \text{ only from the B.L. Theory}\right)$$

$$④ \;=\mu \left(\cancel{\frac{\partial u}{\partial y}}\right)_{y=\delta} - \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = -\,\tau_0$$

Therefore, the B.L. equation becomes

$$\frac{\partial}{\partial x}\int_0^{\delta} \rho\, u^2\, \mathrm{d}y - \rho_{\mathrm{e}} u_{\mathrm{e}}^{2} \frac{d\delta}{dx} - u_{\mathrm{e}} \frac{\partial}{\partial x}\int_0^{\delta} \rho\, u\, \mathrm{d}y + \rho_{\mathrm{e}} u_{\mathrm{e}}^{2} \frac{d\delta}{dx} + \rho_0 u_0$$

$$= -\left(\frac{dp}{dx}\right)\delta - \tau_0 \tag{6.4}$$

If we evaluate the B.L. equation at $y = \delta$, we have

$$\left\{\rho\left[u_{\mathrm{e}}\frac{\partial u}{\partial x} + v_{\mathrm{e}} \underbrace{\cancel{\frac{\partial u}{\partial y}}}_{=0}\right] = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left[\mu \underbrace{\cancel{\frac{\partial u}{\partial y}}}_{(\,=0 \text{ at } y=\delta\,)}\right]\right\}_{\text{at } y=\delta}$$

$$\rho\, u_{\mathrm{e}} \frac{du_e}{dx} = -\frac{dp}{dx} \tag{6.5a}$$

Also

$$u_{\mathrm{e}} \frac{\partial}{\partial x}\int_0^{\delta} \rho\, u\, \mathrm{d}y = \frac{\partial}{\partial x}\left[u_{\mathrm{e}}\int_0^{\delta} \rho\, u\, \mathrm{d}y\right] - \frac{du_e}{dx}\int_0^{\delta} \rho\, u\, \mathrm{d}y$$

$$= \frac{\partial}{\partial x}\left[\int_0^{\delta} \rho\, u\, u_{\mathrm{e}}\, \mathrm{d}y\right] - \frac{du_e}{dx}\int_0^{\delta} \rho\, u\, \mathrm{d}y \tag{6.5b}$$

Sub. (6.5a) & (6.5b) into (6.4), we have

$$\frac{\partial}{\partial x}\int_0^{\delta} (\rho\, u^2 - \rho\, u\, u_{\mathrm{e}})\mathrm{d}y + \frac{du_e}{dx}\int_0^{\delta} \rho\, u\, \mathrm{d}y - \rho_{\mathrm{e}} u_{\mathrm{e}} \frac{du_e}{dx}\delta + u_{\mathrm{e}}\rho_0 v_0 = -\tau_0$$

or

$$\frac{d}{dx}\rho_{\mathrm{e}} u_{\mathrm{e}}^{2} \int_0^{\delta} \frac{\rho u}{\rho_e u_e}\left(1 - \frac{u}{u_e}\right)\mathrm{d}y + \rho_{\mathrm{e}} u_{\mathrm{e}} \frac{du_e}{dx}\int_0^{\delta}\left[1 - \frac{\rho u}{\rho_e u_e}\right]\mathrm{d}y - u_{\mathrm{e}}\rho_0 v_0 = -\tau_0$$

Lf we define

$$\text{Displacement thickness} \equiv \delta^* \equiv \int_0^\delta [1 - \frac{\rho u}{\rho_e u_e}] dy$$

$$\text{Momentum thickness} \equiv \theta \equiv \int_0^\delta \frac{\rho u}{\rho_e u_e}(1 - \frac{u}{u_e}) dy \tag{6.6}$$

Remark: for incompressible flow, $\rho_e = \rho$, the definition of δ^* and θ is the same as those in the previous chapter.

Then the equation becomes

$$\boxed{\tau_0 = \frac{d}{dx}(\rho_e u_e^2 \theta) + \rho_e u_e \frac{du_e}{dx}\delta^* - u_e \rho_0 v_0} \tag{6.7}$$

"Karman's Integral Momentum Relation"

Remark:

(1) For a given problem, $\rho_e(x)$, $u_e(x)$, ρ_0, v_0 are known therefore, we have three unknown δ^*, θ and τ_0, but has only one equation. How can we solve the equation?

(2) For an incompressible flow ($\rho = \rho_e =$ const), and $\rho_0 = \rho_e$ the integral momentum equation becomes

$$\frac{\tau_0}{\rho} = u_e^2 \frac{d\theta}{dx} + (2\theta + \delta^*) u_e \frac{du_e}{dx} - u_e v_0 \tag{6.8a}$$

or in dimensionless form

$$\boxed{\frac{C_f}{2} = \frac{d\theta}{dx} + \frac{1}{u_e}\frac{du_e}{dx}(2\theta + \delta^*) - \frac{v_0}{u_e}} \tag{6.8b}$$

where

$$C_f = \frac{\tau_0}{\frac{1}{2}\rho u_e^2}$$

6.2 Solution of the Integral momentum equation

If we assume a non-dimensional shape of the velocity profile, such as

$$\frac{u}{u_e} = f\left(\frac{y}{\delta}\right) \tag{6.9}$$

then Eqn(6.7) will reduced to one equation for one unknown $\delta(x)$, since δ^*, θ can be obtained by integrating the assumed velocity profile. We try a simple problem to see whether this ideal work or not. (and τ_0 can be obtained by set $\tau_0 = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}$)

Consider a incompressible flow past a flat plate without suction / injection, then Eq.(6.8) becomes ($du_e/dx = 0$)

$$\frac{d\theta}{dx} = \frac{C_f}{2} = \frac{\tau_0}{\rho u_e^{\,2}} \tag{6.10}$$

The velocity profile must satisfy $u(0) = 0$ (No slip wall condition) and $u(\delta) = u_e$. Take the simple guess for the velocity profile, we assume

$$\frac{u}{u_e} = \frac{y}{\delta} \tag{6.11}$$

then
$$\theta = \int_0^{\delta} \frac{u}{u_e}\left(1-\frac{u}{u_e}\right)dy = \frac{\delta}{6}$$

$$\tau_0 = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu u_e / \delta$$

Therefore, equation (6.10) becomes

$$\boxed{\delta \frac{d\delta}{dx} = \frac{6(\mu/\rho)}{u_e}}$$

integrate once

$$\delta^2 = \frac{12(\mu/\rho)x}{u_e} + \mathrm{C}$$

Since $\delta\,(x = 0) = 0 \qquad \Rightarrow \;\; C = 0$

The boundary layer thickness is thus

$$\delta\,(x) = \sqrt{\frac{12(\mu/\rho)x}{u_e}}$$

or $$\boxed{\frac{\delta}{x}} = \sqrt{\frac{12\mu}{\rho u_e x}} = \boxed{3.46\; Re_x^{-1/2}} \qquad (6.12a)$$

The friction coefficient C_f is

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho u_e^2} = \frac{\mu u_e/\delta}{\frac{1}{2}\rho u_e^2} = \boxed{0.577\; Re_x^{-1/2}} \qquad (6.12b)$$

From the exact solution as shown in chapter 5, we have obtained

$$\frac{\delta}{x} = 5\; Re_x^{-1/2} \qquad (5.4)$$

and $$C_f = 0.664\; Re_x^{-1/2} \qquad (5.5a)$$

Thus, this simple analysis has achieved the correct dependence on Re_x, bit fairly good numerical values for the coefficients.

Question: Any better velocity profiles or better methods?

6.2.1 The Pohlhausen Method (1921)

Pohlhausen assume

$$\frac{u}{u_e} = a + b\,(\frac{y}{\delta}) + c\,(\frac{y}{\delta})^2 + d(\frac{y}{\delta})^3 + e(\frac{y}{\delta})^4 \qquad (6.13)$$

a, b, c, d, e, which may be the function of *x*, are determined by the following B.C's:

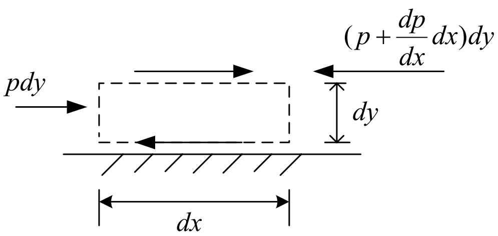
(i) At $y = 0$, $u = 0$ (No slip wall condition)

(ii) At $y = \delta$, $u = u_{\mathrm{e}}$

(iii) At $y = \delta$, $\frac{\partial u}{\partial y} = 0$ (continuous of u at $y = \delta$)

(iv) At $y = \delta$, $\frac{\partial^2 u}{\partial y^2} = 0$

(v) A t $y = 0$, $\mu \frac{\partial^2 u}{\partial y^2} = \frac{dp}{dx}$



$u = v = 0$ near the wall, thus, the momentum flux is negligible.

In equilibrium, pressure forces = shear force.

$$p(\mathrm{d}y) - (\,p + \frac{dp}{dx}\,\mathrm{d}x\,)\mathrm{d}y = -\mu\frac{\partial^2 u}{\partial y^2}\,\mathrm{d}x + \mu\left[\frac{\partial u}{\partial y} + \frac{\partial}{\partial y}(\frac{\partial u}{\partial y})dy\right]\mathrm{d}x$$

$$\Rightarrow \quad \frac{dp}{dx} = \mu\frac{\partial^2 u}{\partial y^2}$$

Note: this is similar to the G.E. for the slowly motion, where the inertia force is neglected too.

$$\Rightarrow \quad \frac{\mu 2 C u_e}{\delta^2} = \frac{dp}{dx}$$

$$\Rightarrow \quad \mathrm{C} = \frac{\chi^2 (dp/dx)}{2\mu\, u_e} = -\frac{\delta^2}{2\nu}\frac{du_e}{dx} \qquad (\frac{dp}{dx} = -u_{\mathrm{e}}\frac{du_e}{dx})$$

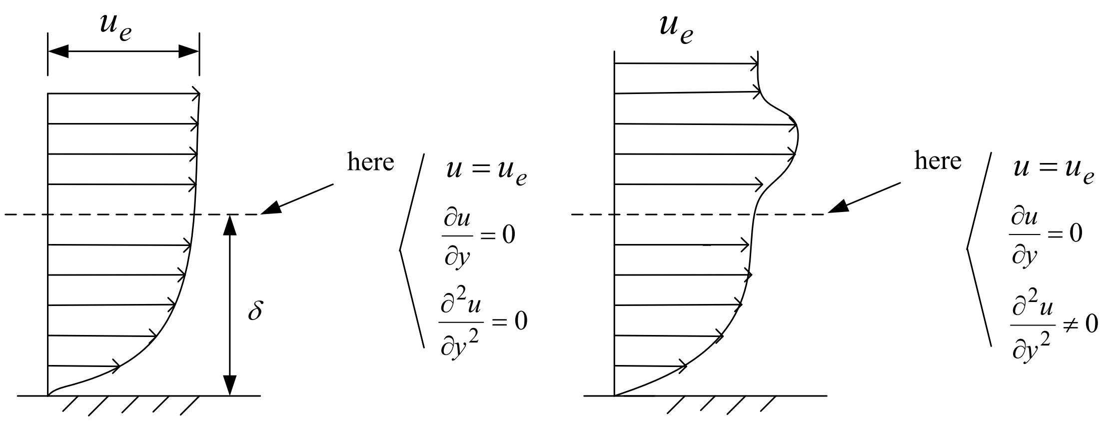
Define:

$$\lambda\,(x) \equiv \frac{\delta^2}{\nu}\frac{du_e}{dx} \equiv \text{Pohlhausen parameter}$$

We have give unknown (*a, b, c, d, e*), but we have 4 B.C's ((i) → (iv)) and define

C = $-\dfrac{\lambda}{2}$ in B.C. (iv); therefore, the *a, b, d, e* can be expressed in terms of λ (*x*). The final results is

$$\begin{cases} a = 0 \\ b = 2 + \dfrac{\lambda}{6} \\ \mathrm{c} = -\dfrac{\lambda}{2} \\ d = -2 + \dfrac{\lambda}{2}, \quad \mathrm{e} = 1 - \dfrac{\lambda}{6} \end{cases}$$



Note:

We assume a velocity profile containing a-e give undetermined coefficient, therefore, we need give B.C's to solve it. The coefficient is expressed in terms of λ, which is dependent on the known potential velocity ($\dfrac{du_e}{dx}$) and a unknown δ (*x*). The δ (*x*) should be determined by the Karman's Integral momentum equation.

Define: $\eta = y/\delta$

The velocity profile becomes

$$\underbrace{\frac{u}{u_e}}_{fn\,(x,\,y)} = \underbrace{(2\eta - 2\eta^3 + \eta^4)}_{\equiv F(\eta)} + \underbrace{\frac{\lambda}{6}}_{fn\,(x)}\underbrace{(\eta - 3\eta^2 + 3\eta^3 - \eta^4)}_{\equiv G(\eta)} \qquad (6.14)$$

Before proceeding to find $\delta(x)$, we first check whether there is some limitation on the value of $\lambda(x)$? (Find $\delta(x)$ or $\lambda(x)$ is equivalent since $\lambda \equiv \frac{\delta^2}{\nu}\frac{du_e}{dx}$ where $\frac{du_e}{dx}$ is known)

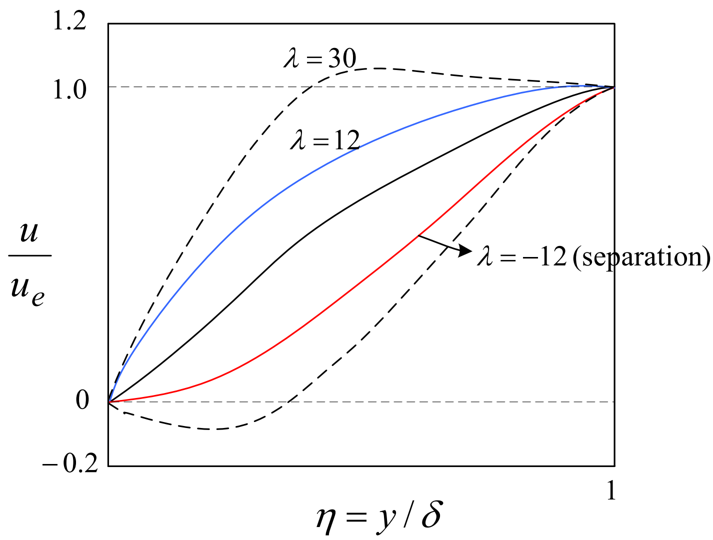
(1) The separation occurs as

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0 \quad \Rightarrow \quad 2 + \frac{\lambda}{6} = 0 \quad \Rightarrow \quad \lambda = -1/2$$

(2) For flow past a flat plate or at a point where u_e reaches its max. or minimum value:

$$\frac{du_e}{dx} = 0 \quad \Rightarrow \lambda = \frac{\delta^2}{\nu}\frac{du_e}{dx} = 0$$

(3) If we plat $u/u_e \sim \eta$ for different value of λ, as shown below:



We find that to maintain u/u_e <<1, it must be $\lambda \le 12$. Therefore, the range of λ is

$$\boxed{-12 \le \lambda \le 12} \qquad (6.15)$$

By the velocity profile given in Eqn (6.14), we get

$$\delta^* = \delta\left(\frac{3}{10}-\frac{\lambda}{120}\right)$$

$$\theta = \frac{\delta}{63}\left(\frac{37}{5}-\frac{\lambda}{15}-\frac{\lambda^2}{144}\right) \tag{6.16}$$

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)\bigg|_{y=0} = \frac{\mu u_e}{\delta}\left(2+\frac{\lambda}{6}\right)$$

Next step is to solve the integral momentum equation in terms of λ (x). For incompressible flow without wall injection/ suction, Eq. (6.8a) gives

$$\frac{\tau_0\theta}{\mu u_e} = \frac{\theta\, u_e}{\nu}\frac{d\theta}{dx} + \left(2+\frac{\delta^*}{\theta}\right)\frac{\theta^2}{\nu}\frac{du_e}{dx} \tag{6.17}$$

Note that equation (6.17) do not contain δ (x) explicitly. We thus try to solve $\theta(x)$, and then deduce δ from it with the cuds of Eq.(6.16).

Introduce

$$Z \equiv \frac{\theta^2}{\nu}$$

$$K \equiv \frac{\theta^2}{\nu}\frac{du_e}{dx} \quad \Rightarrow \quad K = Z\frac{du_e}{dx} = \left(\frac{\theta}{\delta}\right)^2 \overbrace{\frac{\delta^2}{\nu}\frac{du_e}{dx}}^{\lambda}$$

$$(6.16)\rightarrow \; = \left(\frac{37}{315}-\frac{\lambda}{945}-\frac{\lambda^2}{9072}\right)^2 \lambda \tag{6.18a}$$

Denote

$$\frac{\delta^*}{\theta} = \frac{\dfrac{3}{10}-\dfrac{1}{120}\lambda}{\dfrac{37}{315}-\dfrac{1}{945}\lambda-\dfrac{1}{9072}\lambda^2} \equiv f_1(K) \quad \text{(shape-factor correlation)} \tag{6.18b}$$

and

$$\frac{\tau_0\theta}{\mu u_e} = \left(2+\frac{\lambda}{6}\right)\left(\frac{37}{315}-\frac{\lambda}{945}-\frac{\lambda^2}{9072}\right) \equiv f_2(K) \tag{6.18c}$$

Also note that

$$\frac{\theta}{\nu}\frac{d\theta}{dx} = \frac{1}{2}\frac{d}{dx}\left(\frac{\theta^2}{\nu}\right) = \frac{1}{2}\frac{dZ}{dx}$$

Eqn (6.17) becomes

$$f_2(K) = \frac{u_e}{2}\frac{dZ}{dx} + (2 + f_1(K))K$$

or

$$u_e \frac{dZ}{dx} = 2f_2(K) - 4K - 2Kf_1(K) \qquad (6.19)$$

Denote:

$$F(K) \equiv 2f_2(K) - 4K - 2Kf_1(K)$$

$$= 2\left(\frac{37}{315} - \frac{\lambda}{945} - \frac{\lambda^2}{9072}\right)\left[2 - \frac{116\lambda}{315} + \left(\frac{2}{945} + \frac{1}{120}\right)\lambda^2 + \frac{2}{9072}\lambda^3\right] \qquad (6.20)$$

(6.18)

Eqn (6.19) thus becomes

$$\frac{dZ}{dx} = \frac{F(K)}{u_e}, \qquad \text{where } K = Z u_e' \qquad (6.21)$$

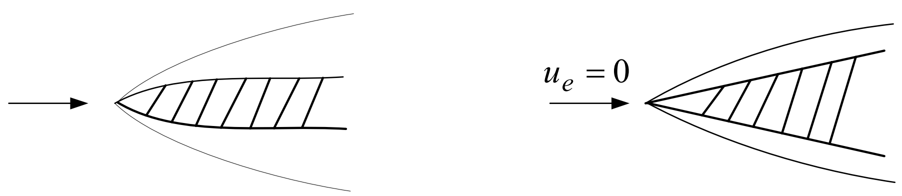
This is a non-linear, 1st order O.D.E for Z as a function of x. It can be solved numerically starting from the initial point. The question is where is the initial point and how large is the initial value?

<u>Initial condition:</u>

The calculation should start at $x = 0$ (stagnation point), where

$$u_e = 0, \quad \frac{du_e}{dx} \neq 0 \text{ but finite value.}$$

(for the flow passing a curved surface body)



Since $$\frac{dZ}{dx} = \frac{F(K)}{u_e}$$

That is, $F(K)$ must be zero at the stagnation point, otherwise, $\frac{dZ}{dx}$ will become infinite, which is physically meaningless. Therefore, at $x = 0$

$$F(K) = 0 \xrightarrow[\text{Eq. (6.20)}]{} \lambda = \lambda_0 = 7.052$$

$$\text{or} \quad K = K_0 = 0.0770$$

The initial value of Z and dZ/dx are

$$Z_0 = \frac{K_0}{\left.\frac{du_e}{dx}\right|_{x=0}} = \frac{0.077}{\left.\frac{du_e}{dx}\right|_{x=0}}$$

$$\left(\frac{dZ}{dx}\right)_0 = \left(\frac{F(K)}{u_e}\right)_0 = \underbrace{\left(\frac{\frac{dF}{dK}\frac{dK}{dx}}{du_e/dx}\right)_0}_{\text{Hosptial Rule}} \underbrace{=}_{(6.20)(6.18a)} -0.0652\frac{\left(\frac{d^2u_e}{dx^2}\right)_{x=0}}{\left(\frac{du_e}{dx}\right)_{x=0}^{2}}$$

<u>Computational procedure:</u>

(1) $u_e(x)$, $\frac{du_e}{dx}$, and $\left.\frac{d^2u_e}{dx^2}\right|_{x=0}$ are given by potential flow.

(2) Integral Eq.(6.21) $\rightarrow Z(x)$ & $K(x)$

(3) By equation of $K = \frac{\theta^2}{\nu}\frac{du_e}{dx}$ $\rightarrow \theta(\text{x})$

(4) By equation (6.18a) $\rightarrow \lambda(x)$

(5) By (6.18b) & (6.18c) $\rightarrow \delta^*$, τ_0

(6) By (6.16) $\rightarrow \delta$

(7) By (6.14) $\rightarrow u/u_e$

The calculation is continuous until $\lambda(x) = -12$ or K = -0.1567, where the separation occurs.

Example: For a flat plate case.

$$\frac{du_e}{dx} = 0 \quad \rightarrow \lambda(x) = \frac{\delta^2}{\nu}\frac{du_e}{dx} = 0$$

The assumed velocity profile (6.14) becomes

don't need this procedure to get $\frac{\delta^*}{\delta}$, $\frac{\theta}{\delta}$, and τ_0

$$\frac{u}{u_e} = 2\eta - 2\eta^3 + \eta^4$$

(6.18) $\Rightarrow$ $K = Z\dfrac{du_e}{dx} = 0$

(6.20) $\Rightarrow$ $F(0) = F(K) = 2(\dfrac{37}{315})(2) = 0.1698$

(6.21) $\Rightarrow$ $\dfrac{dZ}{dx} = \dfrac{F(K)}{u_e} = \dfrac{0.4698}{u_e}$

$$\therefore\ Z = \frac{0.4698}{u_e}x + C$$

Since $x = 0, Z = 0$ (why?) $\rightarrow$ since shape edge flat plate ($\dfrac{du_e}{dx} = 0$), and $Z = \dfrac{\delta^2}{\nu}$), at $x = 0$, $\delta = 0$ $\therefore$ $Z = 0$ at $x = 0$.

$$\therefore\ Z = \frac{0.4698}{u_e}x = \ldots..$$

From (6.16) with $\lambda = 0$, it yields

$$\frac{\delta^*}{\delta} = 0.3, \quad \frac{\theta}{\delta} = \frac{1}{63}(\frac{37}{5}) = 0.1174, \quad \tau_0 = \frac{2\mu u_e}{\delta} \qquad (6.22)$$

From exact solution, we know

$$\delta = 5\sqrt{\frac{\nu x}{u_e}}, \qquad \delta^* = 1.7208\sqrt{\frac{\nu x}{u_e}}, \qquad \theta = 0.664\sqrt{\frac{\nu x}{u_e}}$$

$$\tau_0 = 0.332\,\mu\, u_e\,(\frac{u_e}{\nu X})^{1/2}$$

Take $\quad \dfrac{u}{u_e} = a + b\,(\dfrac{y}{\delta}) + c\,(\dfrac{y}{\delta})^2 + \ldots..$

With $\lambda = 0$, ($\frac{du_e}{dx} = 0$), we have

$$\frac{\delta^*}{\delta} = 0.3, \qquad \frac{\theta}{\delta} = 0.1174, \qquad \tau_0 = \frac{2\mu u_e}{\delta}$$

Sub. into the Karman Integral equation

$$\frac{\tau_0 \theta}{\mu u_e} = \frac{\theta u_e}{\nu}\frac{d\theta}{dx}$$

$$\left(\frac{2\mu u_e}{\delta}\right)(0.11748)\left(\frac{1}{\mu u_e}\right) = (0.11748\,\delta)\left(\frac{u_e}{\nu}\right)\frac{d}{dx}[0.11748]$$

$$\Rightarrow \quad 0.2348 = (0.0138)\frac{u_e}{\nu}\delta\frac{d\delta}{dx}$$

$$\Rightarrow \quad 17.015\frac{\nu}{u_e} = \delta\frac{d\delta}{dx} = \frac{1}{2}\frac{d(\delta^2)}{dx}$$

$$\Rightarrow \quad \frac{d(\delta^2)}{dx} = 34.03\frac{\nu}{u_e}$$

$$\Rightarrow \quad \delta^2 = 34.03\frac{\nu x}{u_e}$$

$$\Rightarrow \quad \boxed{\delta = 5.83\sqrt{\frac{\nu x}{u_e}}} \text{ or } \boxed{\frac{\delta}{x} = 5.83\, Re_x^{-1/2}}$$

And the exact solution is $\delta = 5\sqrt{\frac{\nu x}{u_e}}$, therefore, the Pohlhausen Method is closed exact solution than taking $\frac{u}{u_e} = \frac{y}{\delta}$ case.

Therefore,

$$\begin{cases} \dfrac{\delta^*}{\delta} = \dfrac{1.7208}{5} = 0.344 \\ \dfrac{\theta}{\delta} = \dfrac{0.664}{5} = 0.1328 \\ \tau_0 = 0.332\,\mu\, u_e\left(\dfrac{5}{\delta}\right) = 1.66\dfrac{\mu u_e}{\delta} \end{cases} \tag{6.23}$$

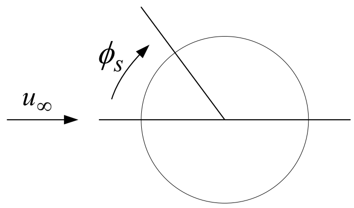
For the simple case with $du_e/dx = 0$. The Pohlhausen's Method is ok. However, for $du_e/dx < 0$, this method becomes some what inaccurate as the point of separation is approached.

For example, for a flow past a circular culinder, the separation point is founded to be as follows.

method	Numerical method slove directly the Differential equation	Blausius series up to x″ term	Pohlhausen's approx. method
ϕ_s	104.5°	108.8°	109.5°

? (Best) (Worse)

(The above result is obtained by calculating $u_e(x)$ from potential flow)



For flow over a flat plate.

Different velocity profiles yield different result

① Assume $u \approx U\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)$ (p.222 Eq. 4-11 in while, we cove flow)

$$\begin{cases} \delta/x \approx 5.5\, Re_x^{-1/2} \\ \delta^*/x \approx 1.83\, Re_x^{-1/2} \\ \theta/x \approx 0.73\, Re_x^{-1/2} \end{cases}$$

② Assume $\frac{u}{U} \approx \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$ (White. Prob.4.1. P.329)

$$\begin{cases} \delta/x \approx 4.64\, Re_x^{-1/2} \\ \delta^*/x \approx 1.74\, Re_x^{-1/2} \\ \theta/x \approx 0.64\, Re_x^{-1/2} \end{cases}$$

③ Assume $\frac{u}{U} \approx sin\,(\frac{\pi y}{2\delta})$ (White. Prob.4.3 p.330)

$$\begin{cases} \delta / x \approx 4.80\, Re_x^{-1/2} \\ \theta / x \approx 0.656\, Re_x^{-1/2} \end{cases}$$

(The B.C's needed for velocity profile is described very completely on p.534 in 吳望一編著，流體力學.（歐亞）)

$$\frac{d}{dx}\int_0^\delta \rho u(U-u)\,dy+\frac{dU}{dx}\int_0^\delta \rho(U-u)\,dy=\tau_w \qquad (9.10.15)$$

(9.10.15)式和(9.10.10)式完全一樣，經過和前面一樣的運算工作再一次得到卡曼動量積分關係式

$$\frac{d\delta^{**}}{dx}+\frac{U'}{U}(2+H)\delta^{**}=\frac{\tau_w}{\rho U^2}$$

由此可以確信(9.10.11)式或(9.10.12)式就是邊界層內動量定理在x方向投影的數學表達。

容易看出，雖然在(9.10.11)式中有三個量δ^*，δ^{**}，τ_w。但當單參數速剖面給出後，三個量中只包含一個未知函數，而(9.10.12)式就是確定參數的常微分方程。

(c) 速度剖面在邊界上應該滿足的條件

在邊界層外部邊界上，粘性流體的速度分量$u(x,y)$應該和位勢外流的速度U相銜接，函數及各級導數都相等。換句話說，要求當$y=\delta$，∞時

$$\begin{cases} u=U\ ,\dfrac{\partial u}{\partial y}=0\ ,\dfrac{\partial^2 u}{\partial y^2}=0 \\ \dfrac{\partial^3 u}{\partial y^3}=0\ ,\cdots,\ \dfrac{\partial^n u}{\partial y^n}=0\ ,\cdots \end{cases} \qquad (9.10.16)$$

這就是速度剖面在邊界層外部邊界上應該滿足的條件。現在進一步考察速度剖面在壁面上應該滿足什麼條件，為此將縱向速度分量$u(x,y)$在壁面$y=0$附近展成泰勒級數

$$u(x,y)=\left(\frac{\partial u}{\partial y}\right)_{y=0}y+\frac{1}{2!}\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}y^2+\frac{1}{3!}\left(\frac{\partial^3 u}{\partial y^3}\right)_{y=0}y^3+\cdots \qquad (9.10.17)$$

這裏已考慮到壁面粘附條件$(u)_{y=0}=0$，將上式代入連續方程得橫向速度$v(x,y)$的泰勒展開式為

$$v(x,y)=\frac{1}{2!}\left(\frac{\partial u}{\partial y}\right)'_{y=0}y^2+\frac{1}{3!}\left(\frac{\partial^2 u}{\partial y^2}\right)'_{y=0}y^3+\frac{1}{4!}\left(\frac{\partial^3 u}{\partial y^3}\right)'_{y=0}y^4+\cdots \qquad (9.10.18)$$

式中"$'$"代表對x的微分：將展式(9.10.17)及(9.10.18)代入邊界層方程的動量方程

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2}\qquad (9.10.1$$

然後令 y 的同冪次係數相等得

$$\begin{cases}\left(\frac{\partial u}{\partial y}\right)_{y=0} & \text{可自由選擇，它是一個參數}\\ \left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}=-\frac{UU'}{\nu} & \text{(a)}\\ \left(\frac{\partial^3 u}{\partial y^3}\right)_{y=0}=0 & \text{(b)}\quad (9.10.2\\ \left(\frac{\partial^4 u}{\partial y^4}\right)_{y=0}=\frac{1}{\nu}\left(\frac{\partial u}{\partial y}\right)_{y=0}\left(\frac{\partial u}{\partial y}\right)''_{y=0} & \text{可自由選擇，亦是一個參數}\end{cases}$$

……

由此可見，速度剖面在壁面上必須滿足（9.10.20(a)）及（9.10.20(b)）等件（將 $y=0$ 時 $u=v=0$ 的條件代入邊界層方程（10.10.19）及它對 y 分後的方程，可以直接得到條件（9.10.20(a)）及（9.10.20(b)）），其中

$$\left(\frac{\partial u}{\partial y}\right)_{y=0},\ \left(\frac{\partial^4 u}{\partial y^4}\right)_{y=0},\ \left(\frac{\partial^7 u}{\partial y^7}\right)_{y=0},\ \cdots\cdots$$

等都是可以自由選擇的參數，其他係數則可通過它們表出。

在壁面上應滿足的條件中，除粘附條件外，當推（9.10.20(a)）最重要它控制速度剖面在順壓區無反曲點，在逆壓區必有反曲點，符合實際情況，因此在曲面物體的繞流問題中，應該盡量使近似速度剖面滿足這個條件，否則就不會有好結果。一般說來，(9.10.16)及(9.10.20)中愈靠前的邊界條件愈重要，應該首先滿足。

d）平板邊界層的近似解

1）速度剖面的選取

平板邊界層具有相似性解，因此 $\frac{u}{U}$ 只依賴於組合變數 $\eta=\frac{y}{\delta}$，即

$$\frac{u}{U}=f(\eta)$$

現在我們選取 $f(\eta)$ 的逼近函數，使它盡量和真實剖面吻合，為此必須盡可能多地滿足邊界上的條件（9.10.16)及（9.10.20)，在平板情形（U＝常數）這些條件可寫成

$$y=0\text{時}\quad u=0,\quad \frac{\partial^2 u}{\partial y^2}=0,\ \frac{\partial^3 u}{\partial y^3}=0,\cdots$$
$$y=\delta\text{時}\quad u=U,\quad \frac{\partial u}{\partial y}=0,\quad \frac{\partial^2 u}{\partial y^2}=0,\cdots \qquad (9.10.21)$$

通常取多項式$\sum_{0}^{n} a_n\eta^n$逼近$f(\eta)$，其中的n個係數a_n可以這樣確定：選取(9.10.21)中最重要的n個邊界條件，令多項式函數滿足它們，得到n個代數方程，把它們解出來即得a_n。下面以一次到四次多項式和三角函數爲例寫出逼近函數。

i) 線性多項式$f(\eta)=a\eta+b$

由$f(0)=0$，$f(1)=1$，定出$a=1$，$b=0$。於是

$$f(\eta)=\eta$$

ii) 二次多項式$f(\eta)=a\eta^2+b\eta+c$

由$f(0)=0$，$f(1)=1$，$f'(1)=0$，定出$a=-1$，$b=2$，$c=0$。於是

$$f(\eta)=2\eta-\eta^2$$

iii) 三次多項式$f(\eta)=a\eta^3+b\eta^2+c\eta+d$

由$f(0)=0$，$f(1)=1$，$f''(0)=0$，$f'(1)=0$，定出$a=-(1/2)$，$b=0$，$c=3/2$，$d=0$。於是

$$f(\eta)=\frac{3}{2}\eta-\frac{1}{2}\eta^3$$

iv) 四次多項式$f(\eta)=a\eta^4+b\eta^3+c\eta^2+d\eta+e$

由$f(0)=0$，$f''(0)=0$，$f(1)=1$，$f'(1)=0$，$f''(1)=0$，定出$a=1$，$b=-2$，$c=0$，$d=2$，$e=0$。於是

$$f(\eta)=2\eta-2\eta^3+\eta^4$$

v) $f(\eta)=\sin\frac{\pi}{2}\eta$

顯然它滿足三次多項式滿足的那些條件。

選定$f(\eta)$的逼近函數，並不是速度剖面就完全確定了，因爲在η中還包含邊界層厚度δ，它是x的函數。當δ依賴於x的函數關係沒有確定以前，我們便不知道在各個不同x截面上應取什麼速度剖面。由此可見，爲了完全確定速度剖面，還需要求出單參數δ和x的關係。

2）單參數 $\delta(x)$ 的確定

確定 $\delta(x)$ 的常微分方程由卡曼動量方程（9.10.11)提供，現在它多
下列形式（U = 常數）

$$\frac{d\delta^{**}}{dx}=\frac{\tau_w}{\rho U^2} \qquad (9.10.2$$

其中

$$\delta^{**}=\int_0^{\delta}\frac{u}{U}\left(1-\frac{u}{U}\right)dy,\quad \tau_w=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \qquad (9.10.2$$

將速度剖面寫成

$$\frac{u}{U}=f(\eta) \qquad (42.10.24$$

其中 $f(\eta)$ 是選定的 η 的已知函數，它的形式可以是 η；$2\eta-\eta^2$；$\frac{2}{3}\eta-\frac{1}{2}\eta$
；$2\eta-2\eta^3+\eta^4$ 或 $\sin(\pi\eta/2)$ 中任一個。將（9.10.24)式代入（9.10
23)式及（9.10.1）式得

$$\delta^{*}=\delta\int_0^1(1-f)\,d\eta=\nu\delta \qquad (9.10.25)$$

$$\delta^{**}=\delta\int_0^1 f(1-f)\,d\eta=\alpha\delta \qquad (9.10.26)$$

$$\frac{\tau_w}{\rho}=\frac{\nu U}{\delta}\left[\frac{\partial(u/U)}{\partial(y/\delta)}\right]_{y=0}=\frac{\nu U}{\delta}f'(0)=\frac{\nu U}{\delta}\beta \qquad (9.10.27)$$

這裏我們令

$$\nu=\int_0^1(1-f)\,d\eta,\ \alpha=\int_0^1 f(1-f)\,d\eta,\ \beta=f'(0) \qquad (9.10.28)$$

它們是完全確定的常數，當 f 的具體形式給出之後，可以根據（9.10.28)式容易地求出它們的數值來。我們將（9.10.26)式及（9.10.27)式代入（9.10.22)式得

$$\alpha\frac{d\delta}{dx}=\frac{\nu}{\delta U}\beta$$

於是確定 $\delta(x)$ 的常微分方程爲

$$\delta\frac{d\delta}{dx}=\frac{\beta\nu}{\alpha U}$$

這個方程非常容易積分，它的解顯然是

$$\delta(x)=\sqrt{\frac{2\beta}{\alpha}}\sqrt{\frac{\nu x}{U}} \qquad (9.10.29)$$

$\delta(x)$ 的形式和準確解結果完全一樣，只是係數略有不同。

3）結果

有了 δ 和 x 的關係（9.10.29），可以確定所有感興趣的物理量，將（9.10.29）代入（9.10.24）得速度剖面為

$$\frac{u}{U}=f\left(\sqrt{\frac{\alpha}{2\beta}}\,y\sqrt{\frac{U}{\nu x}}\right)$$

局部摩阻根據（9.10.27）為

$$\tau_w=\sqrt{\frac{\alpha\beta}{2}}\mu U\sqrt{\frac{U}{\nu x}}$$

於是作用在長為 L 寬為 b 的一段平板上的總摩阻為

$$W=2b\int_0^L\tau_w\,dx=2b\sqrt{2\alpha\beta}\sqrt{\mu\rho LU^3}$$

總摩擦阻力係數為

$$C_f=\frac{W}{2bL\cdot\frac{1}{2}\rho U^2}=\frac{2\sqrt{2\alpha\beta}}{\sqrt{Re}} \qquad (9.10.30)$$

依照（9.10.25）及（9.10.26）可定出排出厚度及動量損失厚度

$$\delta^*=\gamma\sqrt{\frac{2\beta}{\alpha}}\sqrt{\frac{\nu x}{U}},\ \delta^{**}=\sqrt{2\alpha\beta}\sqrt{\frac{\nu x}{U}} \qquad (9.10.13)$$

最後我們將逼近函數選為一次到四次多項式及三角函數 $\sin\frac{\pi}{2}\eta$ 時所計算出來的結果列表如下。

$f(\eta)$	α	γ	β	$\delta\sqrt{\frac{U}{\nu x}}$	$\delta^*\sqrt{\frac{U}{\nu x}}$	$\frac{\tau}{\mu U}\sqrt{\frac{\nu x}{U}}$	$C_f\sqrt{Re}$
η	$\frac{1}{6}$	$\frac{1}{2}$	1	3.46	1.732	0.289	1.155
$2\eta-\eta^2$	$\frac{2}{15}$	$\frac{1}{3}$	2	5.48	1.825	0.365	1.460
$\frac{3}{2}\eta-\frac{1}{2}\eta^3$	$\frac{39}{280}$	$\frac{3}{8}$	$\frac{3}{2}$	4.64	1.740	0.323	1.292
$2\eta-2\eta^3+\eta^4$	$\frac{37}{315}$	$\frac{3}{10}$	2	5.83	1.752	0.343	1.372
$\sin\frac{\pi}{2}\eta$	$\frac{4-\pi}{2\pi}$	$\frac{\pi-2}{\pi}$	$\frac{\pi}{2}$	4.79	1.742	0.327	1.310
準確解				5	1.720	0.332	1.328

上表說明，和準確解相比較，積分關係式方法一般說來給 令人滿意的結果。除線性分布及二次函數外，阻力結果相當精確，與準確解比較誤差不超過3％。

通過平板邊界求近似解，我們了解到利用積分關係式方法求邊界層方程解的主要步驟，同時也初步體會到這個方法的優點，計算簡單，且具有一定的準確度。下面我們進一步利用此方法處理更複雜的曲面邊界層問題。

e）曲面物體邊界層的近似解

1921年波爾豪森利用卡曼動量積分方程處理了具有壓力梯度的曲面物體邊界層問題，他選取四次多項式逼近眞實的速度剖面，即

$$\frac{u}{U}=f(x,\eta)$$

$$=a(x)+b(x)\eta+c(x)\eta^2+d(x)\eta^3+e(x)\eta^4 \qquad (9.10.32)$$

其中a,b,c,d,e是待定的係數，由於曲面物體的邊界層一般說來沒有相似性解，所以u/U不僅依賴於η而且還和x有關，因此a,b,c,d,e都是x的函數，爲了使（9.10.32）盡量和實際剖面接近，波爾豪森令（9.10.32）滿足下列五個邊界條件

$$\begin{cases} y=0\text{時} & u=0\ ,\ \dfrac{\partial^2 u}{\partial y^2}=-\dfrac{UU'}{\nu} \\ y=\delta\text{時} & u=U\ ,\ \dfrac{\partial u}{\partial y}=0\ ,\ \dfrac{\partial^2 u}{\partial y^2}=0 \end{cases} \qquad (9.10.33)$$

由此定出五個函數，在這些條件中，當推$y=0$時

$$\frac{\partial^2 u}{\partial y^2}=-\frac{UU'}{\nu}$$

爲最重要。一切有梯度的物體都必須滿足該條件，它反映了速度剖面的順壓區沒有反曲點，在逆壓區必有反曲點的性質。（9.10.33）式中的第二組條件說明邊界層內的速度剖面和外流的速度剖面在邊界層邊界上二階密切。

將（9.10.32）式代入（9.10.33）式，經過簡單運算後得到a,b,c,d,e滿足的下列方程

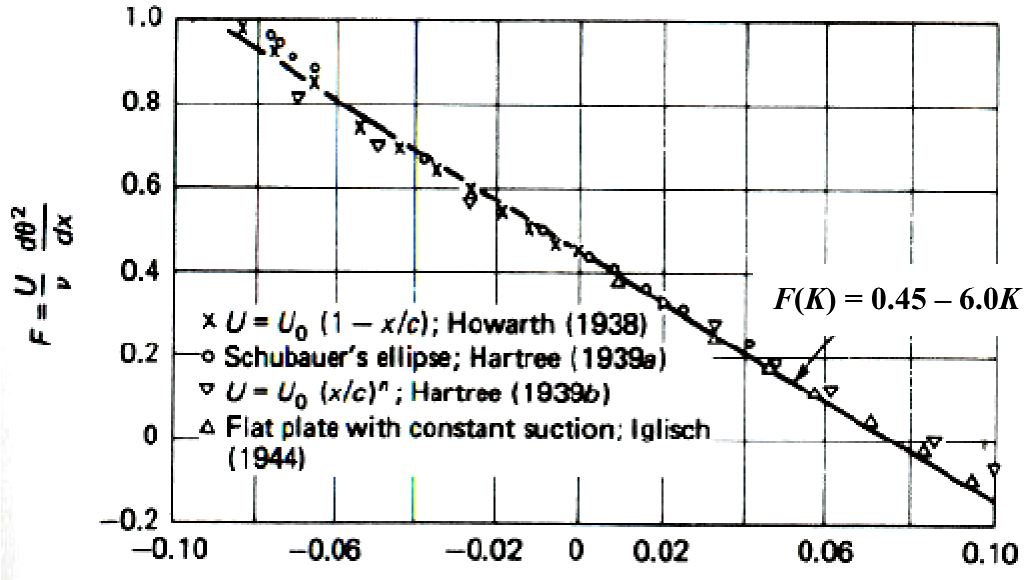
$$\begin{cases} a=0 \\ c=-\dfrac{1}{2}\dfrac{U'\delta^2}{\nu} \\ a+b+c+d+e=1 \\ b+2c+3d+4e=0 \end{cases} \qquad (9.10.34)$$

6.2.2 The Thwaite-Walz Method (1949)

Eq. (6.21) say

$$\frac{dZ}{dx} = \frac{F(K)}{u_e}, \quad K = Z u_e'$$

Thwaits-Walz plat the $F(K) \sim K$ from the Pohlhausen profile and other experimental data, and find that the corresponding curve can be approximated by the formula



K (F. 4-22 on p.269 of White)

$$\boxed{F(K) = 0.45 - 6.0K} \tag{6.24}$$

Therefore

$$u_e \frac{dZ}{dx} = 0.45 - 6.0K$$

$$\underbrace{u_e \frac{d}{dx}\left(\frac{K}{du_e/dx}\right) + 6K u_e^5}_{= \frac{d}{dx}\left(\frac{K u_e^6}{du_e/dx}\right)} = 0.45\, u_e^5$$

$$\Rightarrow \qquad \frac{K u_e^6}{du_e/dx} = 0.45 \int_0^x u_e^5\, dx + C_1$$

Since $u_e(0) = 0 \rightarrow C_1 = 0$

Thus

$$\boxed{K = \frac{0.45}{u_e^{6}} \frac{du_e}{dx} \int_0^x u_e^{5}\, \mathrm{d}x} \tag{6.25}$$

Since $\quad K = \dfrac{\theta^2}{\nu} \dfrac{du_e}{dx}$

$$\therefore \quad \boxed{\theta = 0.45\, \nu\, u_e^{-6} \int_0^x u_e^{5}\, \mathrm{d}x} \tag{6.26}$$

Calculating procedure:

(1) Known $u_e\,(x) \xrightarrow{(6.25)} K(x)$

$\xrightarrow{(6.26)} \theta\,(x)$

(2) By Eqn (6.18b) & (6.18c) $\rightarrow$ (6.27a-d)

$$\begin{cases} \delta^* = \theta f_1(K) \\ \tau_0 = \dfrac{\mu u_e}{\delta} f_2(K) \end{cases}$$

The f_1(k) and f_2(k) are empirical correlated by Thwaits and listed in Table 4.7 on p.314 of White's:"Viscous flow"

k	$f_1(k)$	$f_2(k)$	$F(k)$
⋮	⋮	⋮	⋮

(In F. White, 2nd ed. the f_1(k) and f_2(k) are listed in p.270 table 4-4. The writer shown that f_1(k) and f_2(k) can be curve pitted by the following equations:

$$f_1(k) \approx (\,K + 0.09)^{0.62}$$

$$f_2(k) \approx 2 + 4.14N - 83.5N^2 + 854N^3 - 3337N^4 + 4576N^5$$

where $\quad N = 0.25 - \lambda$

Remark: (1) Thwaites method is about ±5% for favorable or mild adverse gradients but may be as much as ±15% near the separation point.

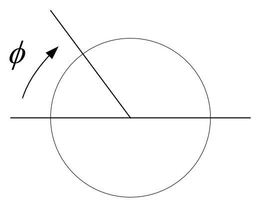
(2) The writer regards Thwaites method as a best available one-parameter method.

(3) If more accuracy is desired FDM is recommended.

Remark:

(1) We previously mention that u_e (x) can be obtained from potential flow. However, in a flow past a blunt body the broad wake caused by bluff – body separation is a first – order effect; i.e., it is so different from potential flow that it alters u_e (x) everywhere, even at the stagnation point. Thus, the potential flow is not suitable input for the boundary – layer calculation. However, once the actual u_e (x) is known, the various theories are exact. For example: for a circular cylinder

	experiment	FDM (Smiths. 1963)	Thwaits	Series method of 19 terms (Howarth)
ϕ_s	80.5°	80.5°	78.5°	83°



Note also the ϕ_s calculated by u_e from the potential flow is about 104 ~109°, which is wrong.

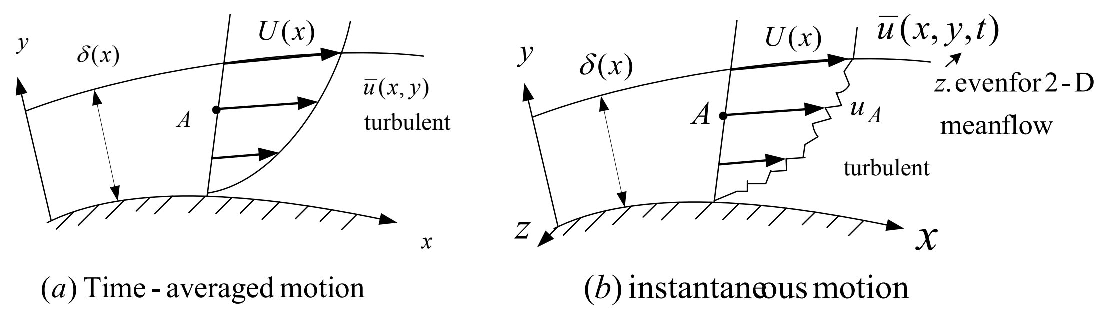
(2) All laminar – boundary – layer calculations hinge upon knowing the correct u_e (x). It is presently a very active area of research to develop coupled methods in which a separating boundary layer interacts with and strongly modifies the external inviscid flow. (e.g.: B.L/shock interaction)

(3) A higher order perturbation method or asymptotic expansion method is applied to match the inviscid & viscous region.

Chapter 7 Turbulent Boundary Layer

<u>Turbulent Boudary Layer Flow</u>

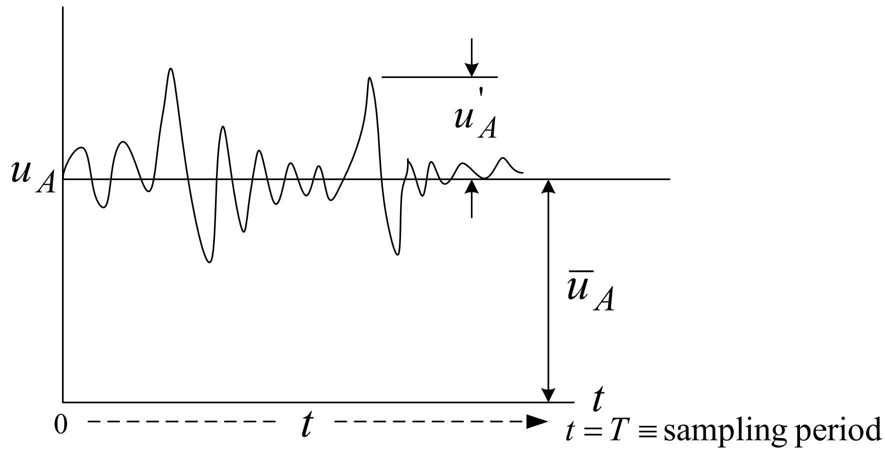
Consider the instantaneous motion in a developing turbulent B.L., as superimposed on the time-averaged or mean motion.



$\overline{u} \equiv$ time–averaged or mean velocity comp. in x – dir.

$u \equiv$ instantaneous velocity comp. in x – dir.

Interrelationship between u & $\overline{u}$:



<u>Definition of time averaged</u> (only work for steady in the mean)

$$\boxed{\overline{(\quad)} = \lim_{T\to\infty} \frac{1}{T}\int_0^T (\quad)\,\mathrm{d}t}$$

At any instant of time:

$$u = \overline{u} + u'$$

Instantaneous mean turbulent fluctuation about mean.

Note:

$$\overline{u} = \lim_{T\to\infty} \frac{1}{T}\int_0^T (\,\overline{u} + u'\,)\,\mathrm{d}t$$

Turbulent is always continuous. Not like shock ware.

$$\overline{u} = \overline{u} + \lim_{T\to\infty} \frac{1}{T}\int_0^T u'\,\mathrm{d}t$$

$$\overline{u} = \overline{u} + \overline{u'} \Rightarrow \boxed{\overline{u'} = 0}$$

from definition.

Steady in the mean.

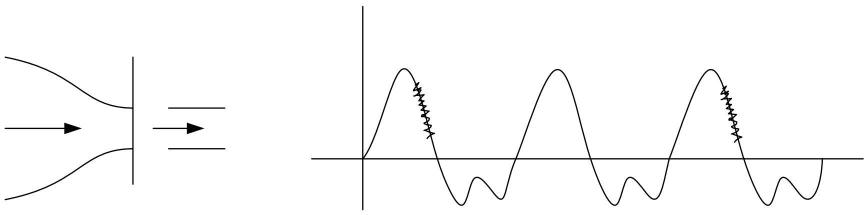
Note:

① The def. of the time averaged applies only for flows that are steady in the mean.

② If the mean motion is unsteady (but periodic), ensemble averaging can be applied to analyze the turbulent.

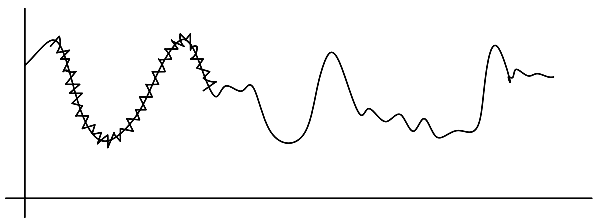
Example:

(a) Periodic mean motion.



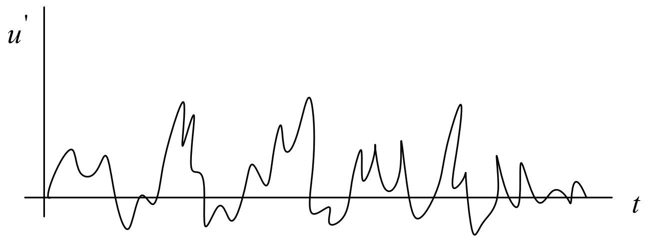
wind tunnel

(b) General unsteady motion



Stay away from this problem

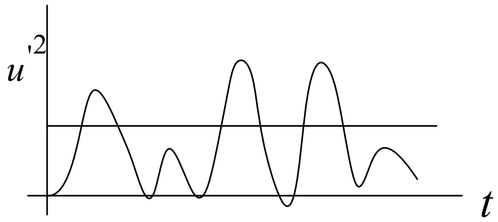
Note: Turbulent fluctuations can be characterized by booking at higher order statistics.



$\overline{u'} = 0$

(a) Velocity – time trace.

(mean value subtracted out)



$\overline{u'^2} \neq 0 \;\rightarrow$ mean aquare

(or $\sqrt{\overline{u'^2}}$ value can be used to charactering the turbulence, e.g.)

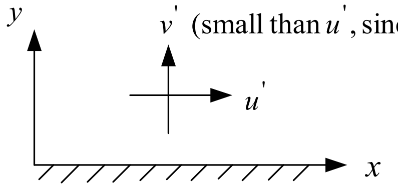
(b) Squared trace.

$$I_{1D} \equiv \frac{\sqrt{\overline{u'^2}}}{\overline{u}} \quad \text{(1-D turbulent intensity)}$$

$$\text{or} \quad I_{3D} = \frac{\sqrt{\frac{1}{3}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})}}{\overline{u}} \quad \text{(3-D turbulent intensity)}$$

$$u',\ v',\ w' \;\Rightarrow\; \overline{u'^2},\ \overline{v'^2}.\ \overline{w'^2} \neq 0$$

v' (small than u', since wall interaction)

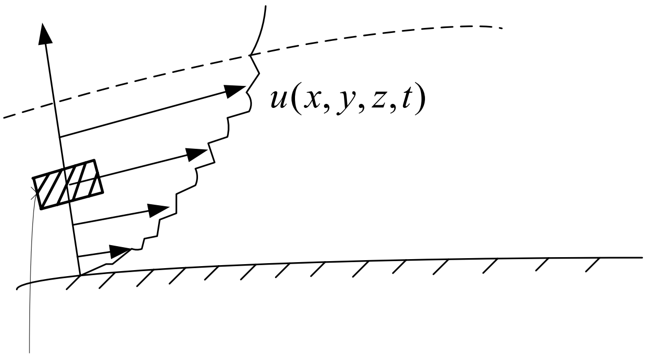


$$\rightarrow \frac{\overline{u'^3}}{\overline{u'^4}} \quad \text{physical meaning}$$

$$\rightarrow \text{diffusion term} \rightarrow \overline{u'v'^2}$$

$$\delta_{ij} \rightarrow \begin{cases} i = j & 1 \\ i \neq j & 0 \end{cases} \qquad \varepsilon_{ijk} \rightarrow \begin{cases} \text{2 equal} = 0 \\ 1 \rightarrow 2 \rightarrow 3 & \rightarrow 0 \\ 1 \rightarrow 3 \rightarrow 2 & \rightarrow -1 \end{cases}$$

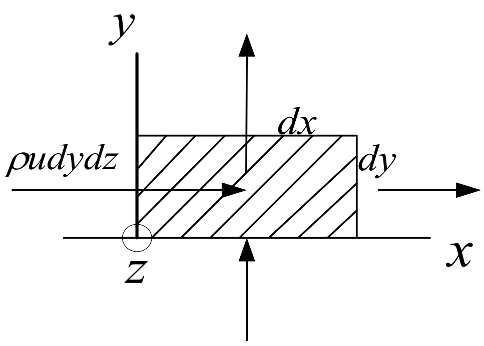
<u>Analyzing turbulent boundary layer flow</u>



Stationary volume element (C.V.)

(a) Physical flow (instantaneous motion)

(b) Mass flux relation to C.V.



$$\rho\, u\mathrm{d}y\mathrm{d}z + \frac{\partial}{\partial x}[(\rho\, u)\mathrm{d}y\mathrm{d}z]\mathrm{d}x$$

Net x – dir mass flux:

$$\frac{d}{dx}(\rho\, u)\mathrm{d}y\mathrm{d}z\mathrm{d}x$$

$\rho = \overline{\rho} + \rho'$	Averaged procedure
$u = \overline{u} + u'$	$\downarrow$
$v = \overline{v} + v'$	Eq. $\rightarrow$ fluctuation time average
$\overline{\rho uv} \Rightarrow \overline{\rho' u' v'}$	Eq. (complicate)

Continuality equation:

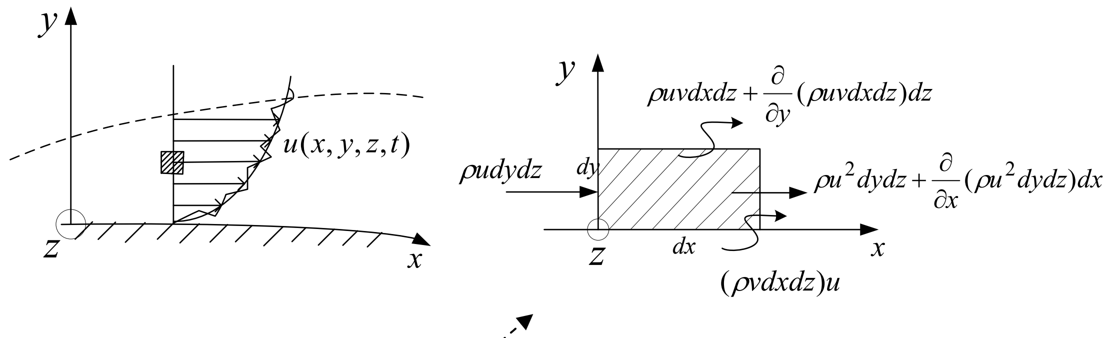
$$\underbrace{\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}}_{\text{complete inst. Mass balance}(x,\, y,\, z)} = 0 \Rightarrow \underbrace{\frac{\partial \rho u_1}{\partial x_1} + \frac{\partial \rho u_1}{\partial x_2} + \frac{\partial \rho u_3}{\partial x_3}}_{(x_1,\, x_2,\, x_3)\text{coordinate system}} = 0$$

$$\Rightarrow \boxed{\frac{\partial(\rho\, u_i)}{\partial x_i} = 0}$$

Steady, Incompressible:

Mass flux consideration $\Rightarrow \boxed{\dfrac{du_i}{dx_i}=0}$ **(1)**

Momentum flux consideration



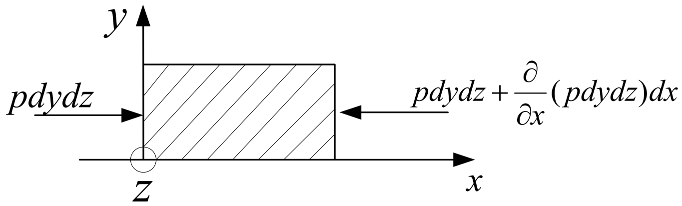
(a) momentum fluxes

$\dot{m} = \rho AV$

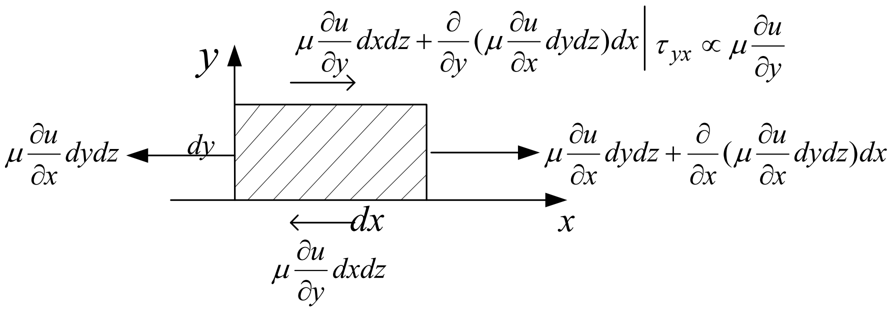
$M = \rho AV^2 = (\dot{m})V$

Balance: Net Momentum efflux from C.V. = sum of all ext. forces acting on fluid in C.V.

(b) pressure force



(c) viscous forces



Balance in x–dir:

$$\frac{\partial}{\partial x}(\rho u^2)(\mathrm{d}x\mathrm{d}y\mathrm{d}z) + \frac{\partial}{\partial y}(\rho uv)\,(\mathrm{d}x\mathrm{d}y\mathrm{d}z) = -\frac{\partial p}{\partial x}(\mathrm{d}x\mathrm{d}y\mathrm{d}z) + \mu\,\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)(\mathrm{d}x\mathrm{d}y\mathrm{d}z)$$

Basic Result (x-dir)

$$\frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \underbrace{\frac{\partial^2 u}{\partial z^2}}_{\text{added term if } z\text{-dir effect analyzed.}}$$

<u>Generalization:</u>

$$\boxed{\frac{\partial \rho\, u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]} \quad \textbf{(2)}$$

(Instantaneous form of momentum equation)

$$\frac{\partial \rho\, u_i u_j}{\partial x_j} = \frac{\partial \rho\, u_1^2}{\partial x_1} + \frac{\partial \rho\, u_1 u_2}{\partial x_2} + \frac{\partial \rho\, u_1 u_3}{\partial x_3}$$

$$= \frac{\partial \rho\, u^2}{\partial x} + \frac{\partial \rho\, uv}{\partial y} + \frac{\partial \rho\, uw}{\partial z}$$

<u>Justification for viscous term</u>:

$$\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] = \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial uj}{\partial x_i}\right)$$

$$= \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right) = \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + \mu \frac{\partial}{\partial x_i}\left(\underbrace{\cancel{\frac{\partial u_j}{\partial x_j}}}_{0\ \text{(continuity eq.)}}\right)$$

for $i = 1$ $$\mu \frac{\partial^2 u_1}{\partial x_1 \partial x_1} + \mu \frac{\partial^2 u_1}{\partial x_2 \partial x_2} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

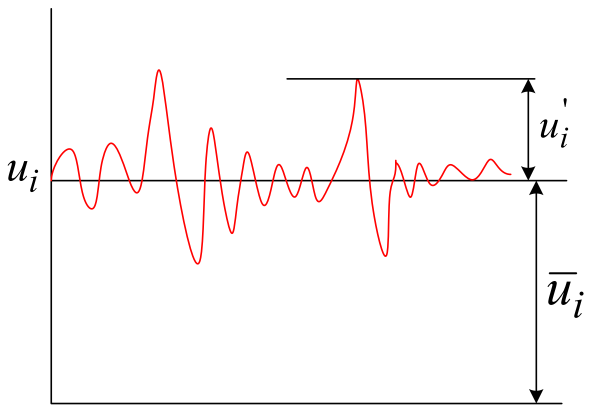
Let:

$$u_i = \overline{u_i} + u_i'$$

$$u_j = \overline{u_j} + u_j'$$

$$p = \overline{p} + p'$$

$$\rho = \overline{\rho} \quad \text{(incompressible)}$$



Definite: $\boxed{\overline{(\quad)} = \lim\limits_{T\to\infty} \dfrac{1}{T}\int_0^T (\quad)\,\mathrm{d}t}$

and time averaged equation (1) & (2), then apply the definite of time averaged quantity.

Consider:

$$\overline{\left[\frac{\partial}{\partial x_i}(\overline{u_i}+u_i^{'})\right]}=0$$

so that $\lim\limits_{T\to\infty} \dfrac{1}{T}\displaystyle\int_0^T \dfrac{\partial}{\partial x_i}(\,\overline{u}_i+u_i^{'}\,)\,\mathrm{d}t$

$$= \frac{\partial}{\partial x_i}[\lim_{T\to\infty}\int_0^T(\,\overline{u}_i\,)\,\mathrm{d}t+\lim_{T\to\infty}\int_0^T(\,u_i^{'}\,)\,\mathrm{d}t]$$

$$= \frac{\partial}{\partial x_i}[\overline{\overline{u_i}}\,]+\frac{\partial}{\partial x_i}\;\underbrace{(\overline{u_i^{'}}\,)}_{=0}$$

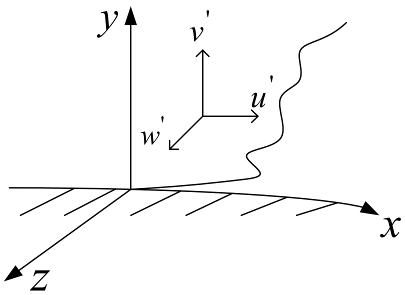
$$= \frac{\partial}{\partial x_i}\overline{u_i^{'}}\ (1)$$

$\Rightarrow \quad \dfrac{\partial u_i}{\partial x_i}=0 \qquad$ continuity equation apply to the mean motion.

Note: $\quad \dfrac{\partial u_i}{\partial x_i}=0 \;\Rightarrow\; \dfrac{\partial}{\partial x_i}(\overline{u_i}+u_i^{'})=0$

or $\quad \dfrac{\partial \overline{u_i}}{\partial x_i}+\dfrac{\partial u_i^{'}}{\partial x_i}=0$

continuity equation applied to the instantaneous fluctuation



$$\boxed{\frac{\partial u^{'}}{\partial x}+\frac{\partial v^{'}}{\partial y}+\frac{\partial w^{'}}{\partial z}=0}$$

Consider now equation (2):

$$\overline{\frac{\partial}{\partial x_j}[\overline{\rho}(\overline{u_i}+u_i^{'})(\overline{u_j}+u_j^{'})]} = -\overline{\frac{\partial}{\partial x_i}(\overline{p}+p^{'})} + \overline{\frac{\partial}{\partial x_j}\left(\mu\frac{\partial(\overline{u_i}+u_i^{'})}{\partial x_j}+\mu\frac{\partial(\overline{u_j}+u_j^{'})}{\partial x_i}\right)}$$

① ② ③

Now

① $\overline{\frac{\partial}{\partial x_j}[\overline{\rho}\overline{u_i}\overline{u_j}+\overline{\rho}u_i^{'}\overline{u_j}+\overline{\rho}u_j^{'}\overline{u_i}+\overline{\rho}u_i^{'}u_j^{'}]}$ =

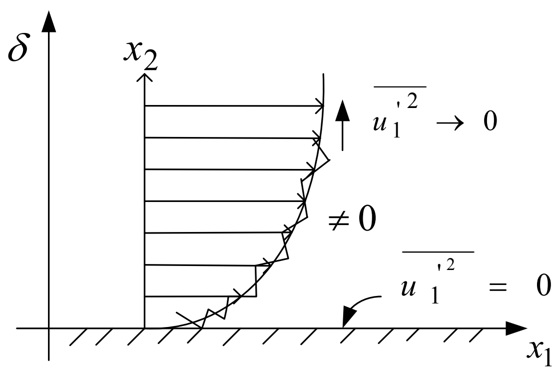
Time averaged of sum is sum.

HW. Give final result on Fr.

Final working result.

Ans: $\overline{\rho}\,\frac{\partial\overline{u_i}\overline{u_j}}{\partial x_j} + \overline{\rho}\,\frac{\partial\overline{u_i^{'}u_j^{'}}}{\partial x_j}$

$\overline{u_1^{'2}} \neq 0 \qquad \frac{\partial\overline{u_1^{'2}}}{\partial x_2} \neq 0$



<u>Consider the instantaneous form of the Momentum Equation</u>

$$\rho\frac{\partial u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\frac{\partial u_i}{\partial x_j} + \mu\frac{\partial u_j}{\partial x_i}\right] \qquad (1)$$

$\rho = \bar{\rho}\,;\ u_i = \overline{u_i} + u_i'\,;\ u_j = \overline{u_j} + u_j'\,;\ p = \bar{p} + p'$

& apply the def. of time average.

$$\bar{\rho}\,\frac{\partial \overline{u_i}\,\overline{u_j}}{\partial x_j} + \bar{\rho}\,\frac{\partial \overline{u_i' u_j'}}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_j} + \frac{\partial}{\partial x_j}\left[\mu\frac{\partial \overline{u_i}}{\partial x_j} + \mu\frac{\partial \overline{u_j}}{\partial x_i}\right]$$

Note: $\dfrac{\partial \overline{u_i u_j}}{\partial x_j} = \overline{u_i}\,\cancelto{0\ (\text{cont.})}{\dfrac{\partial \overline{u_j}}{\partial x_j}} + \overline{u_j}\,\dfrac{\partial \overline{u_i}}{\partial x_j} = \overline{u_j}\,\dfrac{\partial \overline{u_i}}{\partial x_j}$

and $\dfrac{\partial \bar{p}}{\partial x_i} = \dfrac{\partial \bar{p}}{\partial x_j}\delta_{ij}$

$$\begin{Bmatrix} \dfrac{\partial \bar{p}}{\partial x_1} \\ \dfrac{\partial \bar{p}}{\partial x_2} \\ \vdots \end{Bmatrix} = \begin{Bmatrix} \overset{1}{\delta_{11}}\dfrac{\partial \bar{p}}{\partial x_1} + \cancelto{0}{\delta_{12}\dfrac{\partial \bar{p}}{\partial x_2}} + \cancelto{0}{\cdots\cdots} \\ \ddots \end{Bmatrix}$$

only diagonal exist.

So that eq.(1) can be rewritten as:

$$\bar{\rho}\,\overline{u_j}\,\frac{\partial \overline{u_i}}{\partial x_j} = \frac{\partial}{\partial x_j}\Big\{\underbrace{\bar{p}\,\delta_{ij}}_{\substack{\text{Pressure}\\ \text{force}\\ \text{effect}}} + \underbrace{\mu\Big(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\Big)}_{\substack{\text{Viscous}\\ \text{(stress) force}\\ \text{effect}}} - \underbrace{\overline{\rho\, u_i' u_j'}}_{\substack{\text{Turb. Stress}\\ \text{effect or}\\ \text{Reynolds stress}}}\Big\}$$

$\rightarrow\ \tau_{t_{ij}}$ (total stress tensor)

Note: $\tau_{t_{ij}} = \tau_{t_{ji}}$ (total stress tensor is symmetric)

Change *ij* in above equation remain same right side.

At this pt. we have 4 equations in 10 unknown.

(1 const. & 3 Mom. Eq.) in

$$\overset{①}{\overline{u_1}},\ \overline{u_2},\ \overline{u_3},\ \overline{p},\ \overset{③}{\overline{u_1'^2}},\ \overline{u_2'^2},\ \overline{u_3'^3},\ \overline{u_1'u_2'},\ \overline{u_1'u_3'},\ \overline{u_2'u_3'}$$

<u>This is the closure problem</u>.

$$\begin{cases} \text{Thermal energy equation} \rightarrow \overline{u_1},\ \overline{u_2},\ \overline{u_3}, T \\ \text{Velocity-temperature correlation} \rightarrow \overline{u_1'T'},\ \overline{u_2'T'},\ \overline{u_3'T'}, \ldots \quad \text{(heat C. Eq.)} \end{cases}$$

Note: Reynolds transport equation.

$$\overline{u_j}\frac{\partial \overline{u_i'u_k'}}{\partial x_j} + \ldots\ldots\ldots = \frac{\partial}{\partial x_j}(\overline{u_i'u_j'u_k'}) + \ldots\ldots$$

27 terms (at worst)

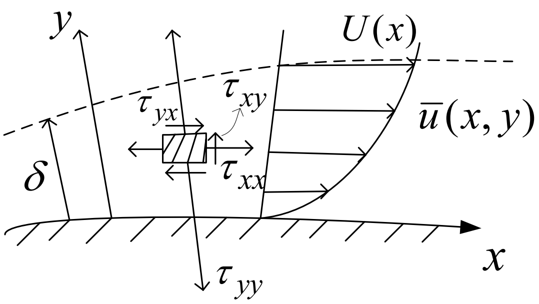
$$\overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} \longrightarrow \text{Convention}$$

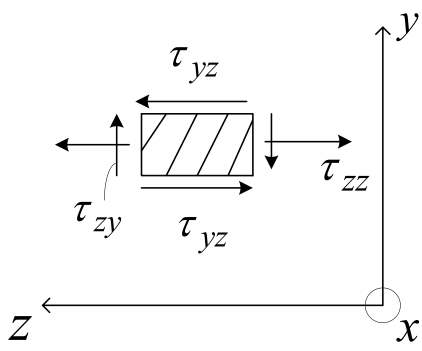
Let $\tau_{ij} = -\rho\,\overline{u_i'u_j'}$

Turbulent Stress tensor

Reduced Form

1. 2-D. b.1. flow in the *xy* plane.

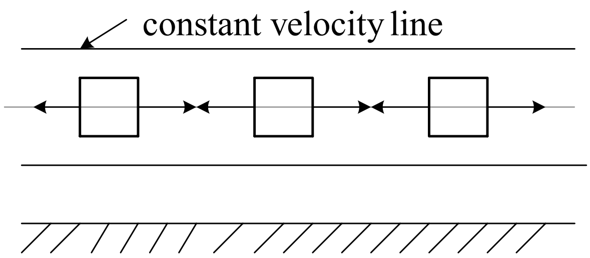




For this case:

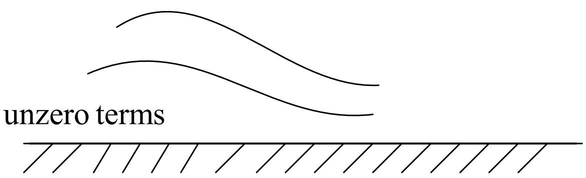
Turbulent is 3-D

$$\left\{\begin{array}{l} \tau_{xx} = -\rho\,\overline{u'^2} \neq 0 \\ \tau_{yy} = -\rho\,\overline{v'^2} \neq 0 \\ \tau_{zz} = -\rho\,\overline{w'^2} \neq 0 \end{array}\right. \qquad \left.\begin{array}{l} \\ \\ \\ \tau_{xy} = -\rho\,\overline{u'v'} \neq 0 \end{array}\right\}$$



$$\tau_{xz} = -\rho\,\overline{u'w'} = 0$$

For unzero terms



$$\tau_{yz} = -\rho\,\overline{v'w'} = 0$$

◎ $\overline{u'^2}$, $\overline{v'^2}$, $\overline{w'^3}$ are all positive.

τ_{xx}, τ_{yy}, τ_{zz} are all neglect (normal stress are compressive.)

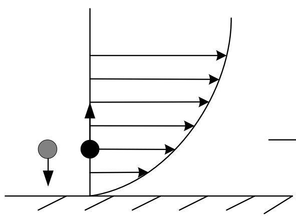
◎ τ_{xy} must positive. $\dfrac{\partial u}{\partial y}$

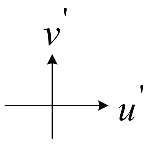
$\rightarrow \overline{u'v'} < 0 \quad \because \boxed{\overline{u'} < 0\ ;\ v' > 0}$

slicting

$v' < 0 \rightarrow$

$u' > 0$





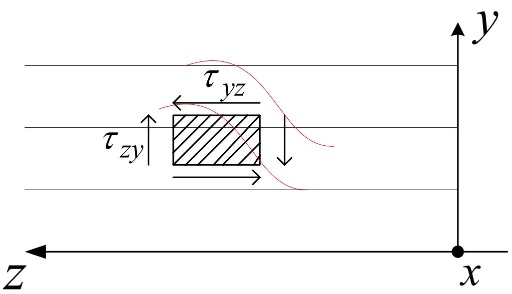
v' is positive, then $u' < 0$.

Positive v' is tend to give a negative u'

$v' =$ positive $\rightarrow u' \rightarrow$ negative

$v' =$ negative $\rightarrow u' \rightarrow$ positive

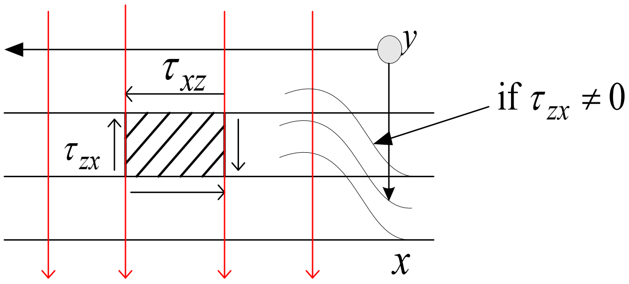
◎ distur. in $\overline{u}$ in yz - plane flow



If τ_{yz} , $\tau_{zy} \neq 0$

$\therefore \tau_{yz} = -\rho\,\overline{v'w'} = 0$

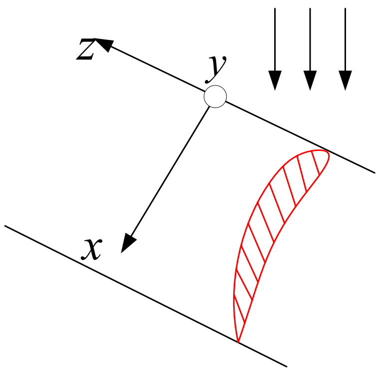
◎ Look down on the flow



$\tau_{zx} = \tau_{xz} = 0$

for 2-D flow.

Example:



Note: For 2-D b.l. flows

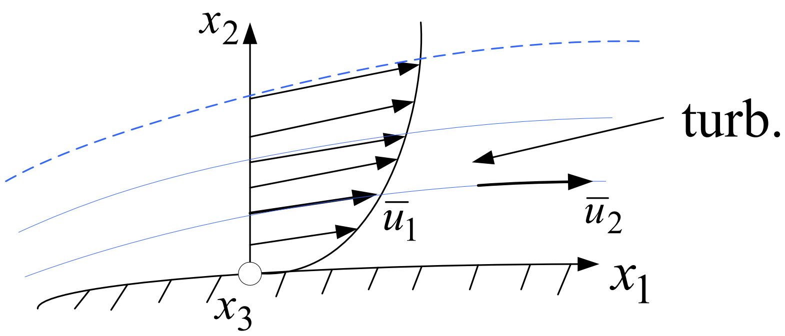
We have 3 equations

6 unknowns

Steady incompressible, 2-D B.L. flow

Boundary Layer Form of the Equations of motion

Consider boundary layer flow in the x_1 - x_2 plane.

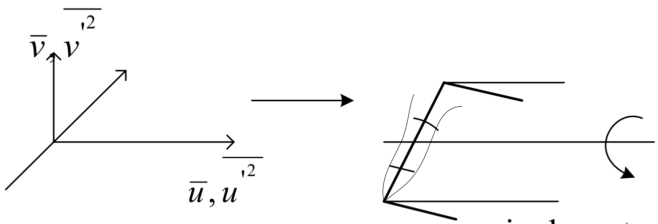


Continuity: $\dfrac{\partial \overline{u}_1}{\partial x_1}+\dfrac{\partial \overline{u}_2}{\partial x_2}=0$ (1)

$$x_1\text{ - }dir\text{ Mom.}: \overline{u}_1\frac{\partial \overline{u}_1}{\partial x_1}+\overline{u}_2\frac{\partial \overline{u}_1}{\partial x_2}= -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_1}+\nu\left(\frac{\partial^2 \overline{u}_1}{\partial {x_1}^2}+\frac{\partial^2 \overline{u}_1}{\partial {x_2}^2}\right)-\frac{\overline{u_1'^2}}{\partial x_1}-\frac{\partial \overline{u_1'u_2'}}{\partial x_2} \quad (2)$$

$$x_2\text{ - }dir\text{ Mom.}: \overline{u}_1\frac{\partial \overline{u}_2}{\partial x_1}+\overline{u}_2\frac{\partial \overline{u}_2}{\partial x_2}= -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_2}+\nu\left(\frac{\partial^2 \overline{u}_2}{\partial {x_1}^2}+\frac{\partial^2 \overline{u}_2}{\partial {x_2}^2}\right)-\frac{\partial \overline{u_1'u_2'}}{\partial x_1}-\frac{\overline{u_2'^2}}{\partial x_2} \quad (3)$$

Note: Eqs (1) ~ (3) have 6 unknowns: $\overline{u}_1$, $\overline{u}_2$, $\overline{p}$, $\overline{u_1'^2}$, $\overline{u_2'^2}$, $\overline{u_1'u_2'}$



wire have to aviod on the wake area

$$x_3\text{ - }dir\text{ Mom.}: 0=0+\nu(0+0)-\cancel{\frac{\partial \overline{u_1'u_3'}}{\partial x_1}}-\cancel{\frac{\partial \overline{u_2'u_3'}}{\partial x_2}}-\frac{\overline{u_3'^2}}{\partial x_3}$$

0 cannot make wringle

0

$\overline{u_3'^2} \neq 0$ Turbulent. 3-D.

but privative $\dfrac{\overline{u_3'^2}}{\partial x_3}=0$

really like to know $\tau_w=\mu\dfrac{\partial \overline{u}_1}{\partial x_2}\Big|_{x_2=0}$

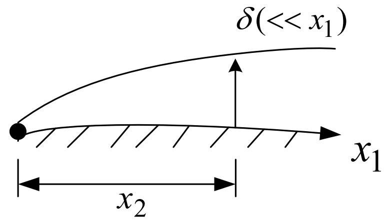
Cannot get $\left.\dfrac{\partial \bar{u}_1}{\partial x_2}\right|_{x=0}$, unless solve the flow field.

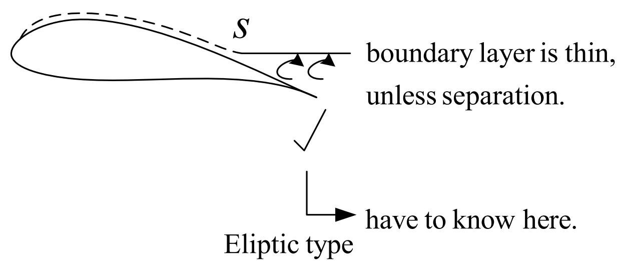
(We need 3 additional equations to effect closure.)

Order – of – magnitude consideration

Let: 1. L_1 & L_2 be length scales in the x_1 & x_2 dir. respectively, ($L_1 \sim x_1$ & $L_2 \sim \delta$)

$\Rightarrow$ $L_2 << L_1$.





2. u_1 & u_2 be velocity scales in the x_1 & x_2 dir. Respectively.

3. V^2 be the velocity scale of each RS comp.

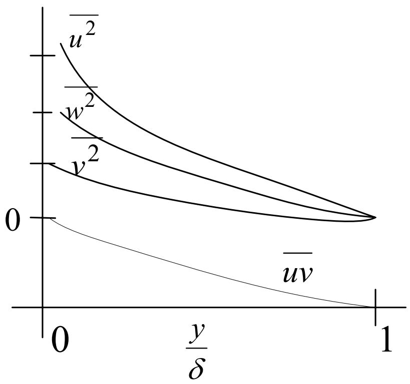
$(\overline{u_i' u_j'})$

$\tau_{ij} = -\rho\, \overline{u_i' u_j'}$

Total stress tensor   Reynolds stress tensor

By correlation: $\overline{u_i' u_j'} \equiv$ "stress" tensor (Kinematics sense)

(Verifiable experimentally)

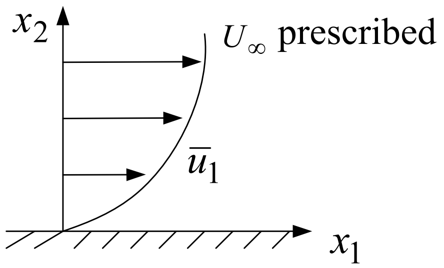


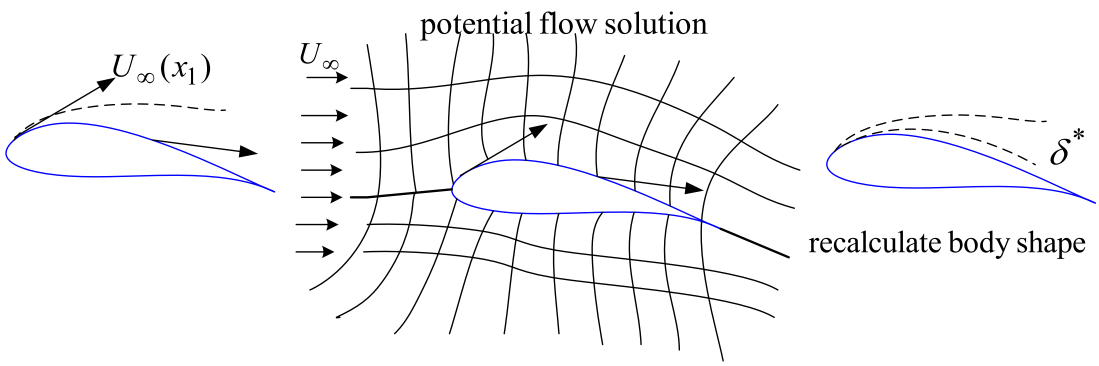
Justification (cross – derivation term)

<u>Assume that</u>

$$\frac{\bar{u}_1}{U_{1(x_1)}} = \left[\frac{x_2}{\delta(x_1)}\right]^{\frac{1}{n}}$$

(a) Laminar case

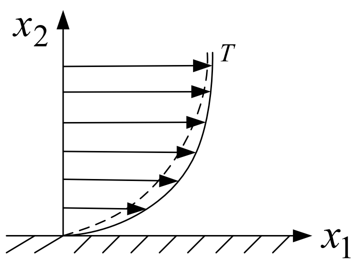




do potential calculation on the original body.

n = 0.5 parabolic $\dfrac{\bar{u}_1}{U_{1(x_1)}} = \left[\dfrac{x_2}{\delta(x_1)}\right]^{2}$

(b) Turbulent flow



($\eta \le$ n ≤ 12) power law.

$$\frac{\bar{u}_1}{U_{1(x_1)}} = \left[\frac{x_2}{\delta(x_1)}\right]^{\frac{1}{7}} \sim \frac{1}{12}$$